The Mathematics Teacher

OCTOBER 1961

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Geometric proof in the eighth grade

MYRON F. ROSSKOPF, *Columbia University, New York, New York.*

Junior high school students should be developing a sensitivity as to what a mathematical proof is.

THERE HAVE BEEN many recommendations in recent years concerning the secondary school geometry program. That the assertion is true could be documented by reference to the publications of a variety of committees, commissions, and projects. One suggestion has been singled out for a closer inspection in this paper. It is that the deductive side of mathematics not be concentrated in one year, the tenth grade traditionally, of Euclidean geometry. Underlying this recommendation is the belief that deduction, or mathematical proof, permeates all of mathematics. Logical deduction in a way *is* mathematics. Certainly the reason that mathematics is a coherent structure is that it is based on what mathematicians mean by a mathematical proof and accept as mathematical proof.

Everyone knows that a mathematical proof of a given statement consists of a sequence of statements, the last one being the statement to be proved. Only certain strictly prescribed statements are eligible to appear in a sequence forming a proof. Each may be an instance of (i) an axiom, (ii) a hypothesis (part of the "Given"), (iii) a definition, (iv) a previously proved theorem, (v) a repetition of an earlier statement in the sequence, (vi) a tautology (valid statement formula), or (vii) it may result from earlier statements in the sequence by means of an inference scheme. It is this last item in the description of a mathematical proof, the use of an inference scheme, that furnishes the "glue" that binds the sequence into an acceptable proof.

The intent of the recommendation to spread instruction in proof over several years is that algebra topics as well as geometry be used as material for teaching purposes. That is, there are as many opportunities for presenting a proof in an algebra setting as there are in a geometry setting—and the axiomatic structure of algebra is much simpler to work with than geometry. But the question of how to do this is largely unanswered. Some tentative steps have been taken to implement a study of proof in an algebra setting in some of the experimental programs, and publications beginning to appear carry the work along. No doubt further steps will be taken as time and classroom experience prove their worth.

However, the whole task cannot be done in one year or two years. Procedures based on accepted principles of learning imply that a foundation be laid several years before a student meets an actual description of a mathematical proof in a classroom and realizes what are the forms of the inference schemes used in a proof. In short, the foundation—or anticipation if you like—ought to be such that a student is likely to say at the time that the formal definitions are given, "Why, that's what we've been doing all along!" It is with this *anticipation* that the present paper deals. Although the setting is an eighth-grade class, it should not be inferred that this is the beginning. The preceding seven years of experiences with mathematics cannot be ignored; the eighth grade is not a fresh, clean slate on which an entirely new program of instruction can be written. It, too, builds on what came before.

Remember this injunction: The specimen problem that follows must be thought

of as occurring after a great deal of work has been done not only in the eighth grade but also in earlier grades. The problem cannot be fairly considered in isolation. A teacher must try to fill in, mentally, what instruction in geometry may have preceded its presentation to the class.

Specimen proposed problem

Let us prove the statement: A diagonal of a parallelogram divides it into two congruent triangles. (See Fig. 1.)

You already know that if two parallel lines are intersected by a transversal, the alternate interior angles are equal in measure. Is $AB \| DC$? Are angles ACD and CAB alternate interior angles formed by these parallel lines and the transversal AC? Then, you can conclude $m\angle ACD = m\angle CAB$. Is $AD \| BC$? Are angles DAC and BCA the alternate interior angles formed by these parallel lines and transversal AC? Then, you can conclude $m\angle DAC = m\angle BCA$. Is the length of AC the same for both triangles ACD and CAB? Do you have two angles and the included side of one triangle equal in measure to two angles and the included side of the other triangle? Then, can you conclude that $\triangle ACD \cong \triangle CAB$?

The problem can be thought of as part of a lesson plan, or even as a problem that might appear on a textbook page. In any case, this problem goes a step beyond what most people understand by "intuitive geometry." Notice that the first sentence serves to recall a fact that must have been discovered earlier and assumed as true for all parallel lines cut by a transversal. This statement, then, plays the role of an axiom. The next question establishes the

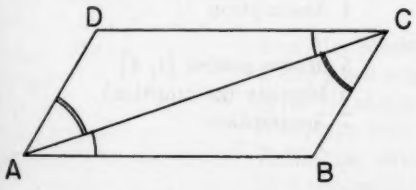


Figure 1

instance of the axiom's "if" part. When the truth of the "if" part has been proved, then the "then" part can be detached and inferred as proved true. You can see that this is an instance of *modus ponens*, or the law of detachment.

Perhaps, the inference scheme called *modus ponens* should be stated:

If "$P \rightarrow Q$" is proved true and
"P" is proved true, then
"Q" can be inferred as proved true.

Then, the above instance displayed schematically is:

$P \rightarrow Q$: If $AB \| DC$ and $\angle ACD$ and $\angle CAB$ are alternate interior angles, then $m\angle ACD = m\angle CAB$.
P: $AB \| DC$ and $\angle ACD$ and $\angle CAB$ are alternate interior angles.
$\therefore Q$: $m\angle ACD = m\angle CAB$.

The next sentences in the problem repeat the same line of reasoning in connection with the parallel lines AD and BC. It presents a situation similar to the first. The chain of statements leading to the conclusion that the two triangles in question are congruent can be displayed as follows:

$R \rightarrow S$: If in triangles ACD and CAB, $m\angle ACD = m\angle CAB$, $mAC = mAC$, and $m\angle DAC = m\angle BCA$, then $\triangle ACD \cong \triangle CAB$.
R: In triangles ACD and CAB, $m\angle ACD = m\angle CAB$, $mAC = mAC$, and $m\angle DAC = m\angle BCA$.
$\therefore S$: $\triangle ACD \cong \triangle CAB$.

There are several points to notice about this proof schema. First, note that it has been simplified for this level by omitting any reference to the definition of a parallelogram. The use of the definition is tacit; it is from it that one can conclude, by use of *modus ponens* again, that the opposite sides of the parallelogram are indeed parallel. Second, it is worth pointing out that there is no mention of *modus ponens*, or an inference scheme, or deduction, in the problem. "Conclude" is used in what might be termed a conversational sense. That is, any eighth-grade student might

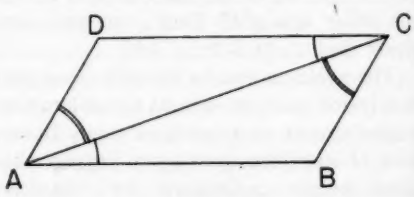
be expected to have this word and an associated meaning as part of his normal vocabulary. So, our use of the term "proof schema" has some merit. We have used a form that a proof should take, but we have not mentioned in any way the names of the various steps, or statements, in the form.

In mathematical logic a demonstration is described in a precise, simple way but at the expense of great length in the demonstrations. For example, an instance of a previously proved theorem is not permitted as a statement in a sequence forming a demonstration. You can see how much length just this one restriction would add to our proofs. To make progress easier we make agreements concerning proofs, or what are called *abbreviated demonstrations*. One such agreement is that instances of previously proved theorems may be used as statements in a proof. The more sophisticated in mathematics one becomes, the more such agreements are practiced. A point is reached at which a proof can be considered a set of clues, or hints, to a demonstration. The clues, with some work on our part, assure us that a demonstration exists; we are satisfied with this assurance and only on very rare occasions even attempt a full demonstration. A secondary-school teacher has the delicate task of bringing his students along through various stages, from an intuitive feeling for proof to a recognition of what a proof is, to proceeding from a large number of clues to a smaller number.

Specimen problem—more formal form

There follows the same parallelogram problem solved in a more formal way. A student who had been acquainted with a definition of a proof and who knew something about inference schemes might write his proof in this way. The form is somewhere between putting a great deal of detail into a proof and merely sketching a proof. There is no particular virtue in the form of this solution. The thought was, simply, that seeing a solution in a more familiar form would serve to emphasize the point of the paper: Any proof that we do at the level of an eighth-grade student ought to anticipate the sort of proof that he will do at a later level of mathematics.

Given: Parallelogram $ABCD$ with diagonal AC; that is, $AB \| DC$ and $AD \| BC$.
To Prove: $\triangle ACD \cong \triangle CAB$.



Statements	*Analysis*
1 Parallelogram $ABCD$; diagonal AC. $AB \| DC$ and $AD \| BC$.	1 Given
2 If $AB \| DC$ and cut by AC, then $m\angle ACD = m\angle CAB$.	2 Assumption
3 $\therefore$ $m\angle ACD = m\angle CAB$.	3 *Modus ponens* [1, 2]
4 If $AD \| BC$ and cut by AC, then $m\angle DAC = m\angle BCA$.	4 Assumption
5 $\therefore$ $m\angle DAC = m\angle BCA$.	5 *Modus ponens* [1, 4]
6 $mAC = mAC$.	6 Identity (assumption)
7 If $m\angle ACD = m\angle CAB$, $mAC = mAC$, and $m\angle DAC = m\angle BCA$, then $\triangle ACD \cong \triangle CAB$.	7 Assumption
8 $\therefore$ $\triangle ACD \cong \triangle CAB$.	8 *Modus ponens* [3, 5, 6, 7]

One objection that can be made to the form of the problem is that it seems to involve much repetition. Perhaps so, but such repetition cannot be avoided if *modus ponens* is to be used correctly. Consider Steps 7 and 8 in the proof. Their form is:

$P \rightarrow Q$: If $m\angle ACD = m\angle CAB$, $mAC = mAC$, and $m\angle DAC = m\angle BCA$, then $\triangle ACD \cong \triangle CAB$.
P: $m\angle ACD = m\angle CAB$, $mAC = mAC$, and $m\angle DAC = m\angle BCA$.
$\therefore Q$: $\triangle ACD \cong \triangle CAB$.

Steps 7 and 8 could be combined in the proof and the use of *modus ponens* made tacit. In case this is done, the step would appear as follows:

7 If $m\angle ACD = m\angle CAB$, $mAC = mAC$, and $m\angle DAC = m\angle BCA$, then $\triangle ACD \cong \triangle CAB$.	7 Assumption
$\therefore \triangle ACD \cong \triangle CAB$.	

The last line might then be preceded by "$\therefore$" as an indication that *modus ponens* had been used. However, when this is done, it should be done consciously as an abbreviation and not considered just that every statement must have a substantiating reason. It is our contention that all too few students in mathematics realize just what the logic of their proofs is. They know they have a proof because they have followed a certain form, but they do not understand why they have a proof that is correct when they follow this form. They possess no criteria for testing whether or not a sequence of statements they have written is a proof. Our claim is that the anticipation of the form in the eighth grade leads to familiarizing students with a correct schema for a mathematical proof that will be explained formally in a later year of study. The task of the teacher making the explanation in this later year will be much easier to perform if his students are already familiar with the correct schema, if they have acquired some sensitivity to what a mathematical proof is.

Mr. B., Mr. V., and Mr. M.

Now Mr. Boole
Was nobody's fool:
He said about axioms: "List 'em!"
And then he came up
With sets, cap and cup—
Model of a Boolean system.

When Mr. Venn
Took up his pen,
He doodled, a habit hodge-podgical.
But the circles he drew
Presented to view
Conclusions remarkably logical.

And Mr. Markov—
He touched a spark off
With his chains and a matrix P.
The steps were the branches
Exhibiting chances
In a probabili-tree.

—*Katharine O'Brien*

Reflections and rotations

BURTON W. JONES, *University of Colorado, Boulder, Colorado.*

Any rigid motion in a plane can be accomplished by three or fewer reflections.

THE SUBJECT of this article is certainly not new to mathematics though its generalization to any number of dimensions and distance functions other than the Euclidean one has been made only in the last twenty years by Dieudonné and others. However, this subject does not seem to be very well known in secondary-school circles and even in undergraduate mathematics, in spite of its fundamental importance. We need assume only the background of a high school course in plane geometry. Any other concepts are defined explicitly.

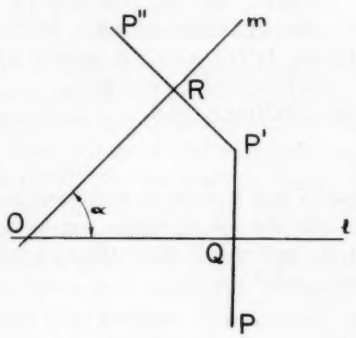


Figure 1

SYMMETRY AND REFLECTIONS

Both within and outside mathematics we are familiar with the idea of symmetry. We know that two points are symmetric with respect to a line if that line is the perpendicular bisector of the line segment joining them. Each of a pair of symmetric points can be thought of as the "mirror image" of the other. Two figures are symmetric in a line if the mirror image of each point of one figure is in the other. A single figure is symmetric in a line if the mirror image of each point of the figure is also in the figure. We can formalize this in a definition.

Definition 1. Given a plane and a line l in that plane, the *mirror image* in l of any point P of the plane is determined as follows:

a) If P is on l, it is its own mirror image.

b) If P is not on l, let t denote the line through P perpendicular to l and intersecting l in the point Q. Then, P', the *mirror image* of P, is the point on t such that Q is the midpoint of the line segment PP'.

Speaking in dynamic rather than in static terms, we may say that P' is obtained by reflecting P in the line l, or that reflection in line l takes P into P'. *Note:* The point P is also the mirror image of P'. If a point P is reflected twice in a line l, it returns to its starting point; that is, a reflection is its own inverse. The only fixed points of a reflection in a line l are those of the line itself.

We have seen what happens when a point is reflected in a line. What happens when a point is reflected first in one of two intersecting lines and then in the other? Consider Figure 1 with two lines l and m intersecting in a point O at an angle α. The image of P in l is denoted by P' and the image of P' in m is denoted by P''. We could continue, reflecting P'' in l and then its image in m, and so forth. It looks as if many points could be obtained in this manner. But they are not located at random all over the plane. Since l is the perpendicular bisector of the line segment PP', P, and P' are equidistant from every point on l and hence, in particular from the point O. Similarly P'' and P' are equi-

distant from O since they are equidistant from every point on line m. Thus P, P', P'' and, indeed, any points obtained from P by any succession of reflections in lines l and m, all lie on the circle whose center is at O and whose radius is the length of OP.

The next thing we notice is that if R is the intersection of $P'P''$ with m, and Q the intersection of PP' with l, the angles POQ and QOP' are congruent, as well as the angles $P'OR$ and ROP''. This shows that the angle POP'' has twice the measure of the angle α.

Thus we have shown, at least for the figure given, that reflecting a point in two successive lines "rotates" the point about the intersection of the lines through an angle which has twice the measure of the angle between the given lines.

Now we give a formal definition of a rotation and a statement of the result we have just discovered, followed by a proof which is less dependent on the figure than the one we have just given.

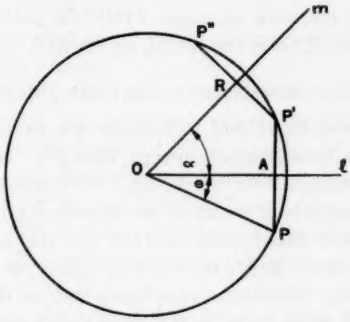


Figure 2

Rotations

Definition 2. Given a point O and an angle β. The point P' is called the image of P under a *rotation* about O through an angle β if this image is determined in the following way:

a) If P is O, its image is itself.

b) If P is not O, P' is the point on the circle with center at O and radius OP such that the measure of the angle, written $m(\beta)$, is equal to $m(\angle POP')$ and in the same direction.

(Actually, a rotation could be *defined* as a succession of two reflections in intersecting lines.) Notice that unless $m(\beta)$ is a multiple of 360°, the only fixed point of this rotation is O. The point O is called the *center of rotation*. If two rotations have the same fixed point and if the measure of their angles differs by a multiple of 360°, we call them *equivalent*, since every point has the same image under one rotation as the other.

If R_1 and R_2 are two rotations about a point O, rotating a point first by R_1 and then by R_2 has the same effect as a rotation about O through an angle whose measure is the sum of the measures of R_1 and R_2. If the sum of the measures of the two angles is a multiple of 360°, we call one rotation the *inverse* of the other. The only rotations which are their own inverses are those the measures of whose angles are multiples of 180°. It can be shown that the rotations about a fixed point form a group, but we do not need this here.

The result which we proved informally we now state as our first theorem.

Theorem 1. A reflection in a line l followed by one in a line m which intersects l in a point O, is equivalent to (that is, has the same effect as) a rotation about O through an angle whose measure is twice that of the angle from l to m and in the same direction. See Figure 2.

Proof: We notice first, as before, that the reflections do not change the distance of the point from O. Hence, we may locate all points obtained by these reflections of P, on the circle with center at O and radius equal to the length OP. We take the line l to be horizontal and call A a point where the circle intersects it. Then if the measure of the directed angle AOP is θ, we may call θ the "co-ordinate of P." Notice that θ may be positive, negative, or zero. The co-ordinate of P' will then be $-\theta$. The directed angle $P'OR$ will have measure $\alpha-(-\theta)=\alpha+\theta$. Hence the measure of angle AOP'' is then $\alpha+\alpha+\theta=2\alpha+\theta$, and

the measure of angle POP'' is $2\alpha+\theta-\theta=2\alpha$. This is the result we wanted.

Transformations and rigid motions

Rotations and reflections are examples of a more general concept, that of a "transformation" or "mapping." By means of a reflection or rotation we take every point of the plane into another (or the same) point of the plane, called its image, in such a way that each point has a unique image and each point is the image of a unique point. Although we think of this in dynamic terms, there is some advantage in a static definition.

Definition 3. A "one-to-one transformation (or mapping) of a plane onto itself" is a correspondence $P \rightarrow P'$ between the points of a plane and the points P' of the same plane such that

a) To each point P of the plane corresponds a unique point P' of the plane, called its image.

b) Each point P' of the plane is the image of a unique point P. We may think of each point P as "transformed," "transported," or "mapped into" another point P'.

If M denotes the transformation described, the correspondence $P' \rightarrow P$ is also such a mapping; it is called the inverse of M and denoted by M^{-1}. If M_1 and M_2 are two such transformations, then M_1 followed by M_2 is also such a transformation and is denoted by M_1M_2 (sometimes by M_2M_1). Two transformations are said to be *equivalent* if they have the same effect on all points of the plane.

It has been said that a geometry is determined by the kinds of transformations which are permitted in it. In other words, a geometry is a study of properties of figures (in plane geometry we would call any set of points and lines a "figure") which remain unchanged by a group of transformations. The transformations which define Euclidean geometry are the so-called rigid motions which we now define.

Definition 4. A "rigid motion in a plane" is a one-to-one transformation of the plane onto itself which preserves distances. That is, if P_1 and P_2 are any two points of the plane and P_1' and P_2' their respective images, then $P_1P_2 = P_1'P_2'$, where P_1P_2 denotes the length of the line segment $\overline{P_1P_2}$. Two figures are said to be *congruent* if one may be taken into the other by a rigid motion. (A more technical term for rigid motion is "isometry.")

Euclidean geometry could be described as a study of properties of figures which remain unchanged under rigid motions. (The rigid motions form a group.) It is not hard to prove the following theorem, hence we do not take the space to prove it here.

Theorem 2. A reflection in a line l is a rigid motion.

Theorem 1, then, immediately gives us the following:

Corollary. A rotation is a rigid motion.

The chief purpose of this article is to give a proof of the remarkable fact that any rigid motion in a plane can be accomplished by three or fewer successive reflections.

Rigid motions and reflections

Theorem 3. If R is a rigid motion in a plane, there is a succession of at most three reflections which is equivalent to it.

We divide the proof into three parts:

Part I. First let A, B, C, be three noncollinear points and A', B', C' their respective images under some rigid motion. We show that A', B', C' are also noncollinear. This is quickly done by noticing that if A', B', C' are collinear, one of the three following distances is equal to the sum of the other two:

$$A'B', \qquad A'C', \qquad B'C'.$$

But this would imply the same statement for the unprimed letters which would deny the fact that A, B, C are taken to be noncollinear. See Figure 3.

Part II. Let ABC and $A'B'C'$ be two triangles in the same plane such that corresponding sides have equal lengths. Then we prove that there is a sequence of at most three reflections which takes the set

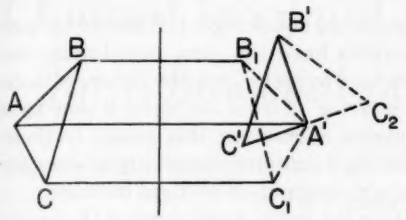


Figure 3

of points A, B, C, into the set A', B', C'.

First, to take A into A' we need merely reflect the set of points A, B, C in the perpendicular bisector of the line segment AA'. (If A and A' coincide, no reflection is necessary.) Call B_1 and C_1 the images of B and C, respectively, under this reflection. Then to take B_1 into B' we should next reflect in the perpendicular bisector of the line segment B_1B'. This will take B_1 into B', but what does it do to A'? The first reflection was a rigid motion and took A into A' and B into B_1 which shows that $AB=A'B_1$. By hypothesis, $AB=A'B'$. Hence $A'B'=A'B_1$ and A' is on the perpendicular bisector of the line segment $B'B_1$. Thus the second reflection leaves A' unchanged. Thus, by two successive reflections we have taken A into A' and B into B'.

Suppose C_2 is the image of C under the two reflections. If $C_2=C'$, the proof is complete. If $C_2\neq C'$, reflect A', B', C_2 in the perpendicular bisector of $C'C_2$. Since $C_2A'=C'A'$ and $C_2B'=C'B'$, both A' and B' are on this perpendicular bisector and hence are not changed by the third reflection.

This completes the proof of Part II.

Since a sequence of reflections is a rigid motion (the rigid motions form a group, and a reflection is a rigid motion), we have, in the process, shown that for any two sets of three points satisfying the requirements of Part II, there is a rigid motion taking either set into the other. In other words, the triples of points are congruent. (We have not yet shown that the complete triangles are.) We now need to show that there is only one rigid motion that does this.

Part III. We prove that there is only one rigid motion which takes the three noncollinear points A, B, C into A', B', C' satisfying the conditions of Part II. To do this, let P be any point different from A, B, C. We need to show that the images of A, B, C under a rigid motion determine the image P' of P.

Since the transformation is a rigid motion, we have the following:

$$AP=A'P', \qquad CP=C'P', \qquad AC=A'C'.$$

Thus P' lies on a circle with center A' and radius equal to AP, and also on a circle with center C' and radius equal to CP. Since $AP+CP\geq AC=A'C'$, the two circles either intersect in two points P' and P'' symmetric with respect to the line $A'C'$, or $AP+CP=AC=A'C'$, and the two circles have just one point in common. In the latter case, P' is on the line $A'C'$ and between A' and C' and is uniquely determined.

It remains to consider the case $AP+CP>AC$, when the two circles have two points of intersection: P' and P''. Now since our transformation is a rigid motion, we have also $BP=B'P'=B'P''$. This means that B' is equidistant from P' and P'' and hence on the line $A'C'$, which is the perpendicular bisector of the line segment $P'P''$. This is impossible by Part I. Thus we have shown that P' is uniquely determined, and Part III is proved.

Now to complete the proof of the theorem, suppose R is any rigid motion and let it take three noncollinear points A, B, C into three points A', B', C'. There is a rigid motion composed of a succession of at most three reflections taking A, B, C, into A', B', C' respectively. Since these images determine the rigid motion, R must be equivalent to some succession of not more than three reflections. This completes the proof.

Notice that it is *not* true that the reflections composing a rigid motion are determined by the rigid motion. For instance,

we might have begun our proof by reflecting about the perpendicular bisector of BB'. But there is just one rigid motion.

In Part III we showed a little more than we needed to show, for we gave a construction of the image of any point P, once the images of A, B, and C are determined. We also showed in the course of the proof that if P lies on a side of a triangle ABC, its image P' lies on the corresponding side of the triangle $A'B'C'$. This implies that if ABC and $A'B'C'$ are two triangles whose corresponding sides are congruent, there is a rigid motion taking one into the other (the sides as well as the vertices), and hence that the two triangles are congruent.

Further remarks

We have discussed what happens if a point is reflected successively in two intersecting lines. The reader might be interested in seeing what happens when the two lines are parallel.

There is also the question of *orientation*, which is a bit elusive if we try to be completely rigorous but which has a simple intuitive meaning. If as we "move around" the triangle ABC (from A to B to C) we "travel" in a counterclockwise direction, we may say that the triangle has counterclockwise orientation. Another triangle, $A'B'C'$ might have clockwise orientation. Intuitively it is not hard to see that a reflection changes the orientation of a triangle. Hence, an odd number of reflections change the orientation and an even number do not change it. If two congruent triangles have the same orientation, one can be "moved" into the other without "removing" it from the plane; if they have opposite orientation, this cannot be done. The rigid motions preserving orientation form a subgroup of all rigid motions.

One can try to avoid some of the logical difficulties inherent in the previous paragraph by *defining* the orientation of two congruent triangles to be the same if one can be taken into the other by an even number of reflections. Then one is faced with the necessity of showing that if one triangle is taken into another by two successions of reflections, then the numbers of reflections in the two successions are both even or both odd. This is quite easy if one sets up a co-ordinate system. The reader might like to try his hand at this.

Also, these methods can be extended without too much trouble to three dimensions. Here one would be concerned with reflections in a plane and rotations in a plane perpendicular to a given axis. In fact, similar results hold in any number of dimensions, but beyond three, the argument must become essentially algebraic even though geometric terms may still be used.

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Have you read?

Bruce, Matthew H., Jr., "Using the Cathode-Ray Oscilloscope in the High School Trigonometry Classroom," *School Science and Mathematics*, November, 1960, pp. 593–602.

It is true that mathematics is an abstract subject and may be studied as such; however, it can often be made more meaningful if we associate the development of an idea with some concrete situation. This article does this by using the oscilloscope to give meaning to the trigonometric functions. Its use demonstrates the periodicity, amplitude, and phase. It also demonstrates the results of combining trigonometric functions. The author gives the specific demonstrations he has found valuable, such as the tuning fork, the alternating current, a microphone and speaking, an organ pipe, and others. If you are teaching trigonometry, you will not want to miss this article. Have all of your class read it. They may add other experiments.—Philip Peak, *Indiana University, Bloomington, Indiana.*

Optimal length of play for a binomial game

FREDERICK MOSTELLER, *Harvard University, Cambridge, Massachusetts.*

In a game where A plays against B, how many trials should A choose to play to maximize his chance of success?

A GAME BETWEEN PLAYERS A and B consists of $N(=2n)$ independent trials. One point is awarded on each trial—to player A with probability $p<\frac{1}{2}$, or to player B with probability $q(=1-p)$. If player A wins more than half the points, he wins a prize. How shall he choose N to maximize his chance of success?* Player A knows the value of p.

At first blush, most people notice that the game is unfair, and therefore that as N increases, the expected value of the difference (A's points−B's points) grows more and more negative. They conclude that A should play as little as he can and still win—that is, two trials.

Had an odd number of trials been allowed, this reasoning based on expected values would have been correct, and A should play only one trial (and this note would not have been written). But with an even number of trials there are two effects at work, (1) the bias in favor of B, and, opposing that at first (2) the redistribution of the probability in the middle term of the binomial distribution (the probability of a tie) as the number of trials increases.

Consider, for a moment, a fair game ($p=\frac{1}{2}$). Then the larger N, the larger A's chance to win because as $2n$ increases, the probability of a tie tends to zero, and in the limit A's chance to win is $\frac{1}{2}$. For $N=2$, 4, 6, his probabilities are $\frac{1}{4}$, $\frac{5}{16}$, $\frac{22}{64}$. Continuity suggests that for p slightly less than $\frac{1}{2}$, A should play a large but finite number of games. But if p is small, $N=2$ should be optimal for A. It turns out that for $p<\frac{1}{3}$, $N=2$ is optimal.

AN APPROXIMATION

Let us use the usual normal approximation to estimate the optimal value of N. Let the random variable X be A's number of points in $N=2n$ trials. We wish to maximize $P(X\geq n+1)$. Let

$$z=\frac{(n+1)-2np-\frac{1}{2}}{\sqrt{2npq}}$$

where the $\frac{1}{2}$ is the usual continuity correction.

Then if Z is a standard normal random variable (zero mean and unit variance), $P(Z>z)$ approximates $P(X>n)$. The smaller z, the larger $P(Z>z)$, so we wish to find the value of n that minimizes z for a fixed $p<\frac{1}{2}$.

Standard calculus methods yield for the value for N that minimizes z

$$N=2n=\frac{1}{1-2p}\,.$$

For example, if $p=0.49$, $N=50$, and, wonder of wonders, this value of N is not just approximate, but is exactly the one that maximizes $P(X\geq n+1)$. Can such good fortune continue? Consider $p=\frac{1}{3}$, then the estimated value of N is 3. It turns out that the surrounding even values of N, 2 and 4, give identical probabilities of a win for A, namely $\frac{1}{3}$; and this $\frac{1}{3}$ is the optimal probability that A can achieve!

* P. G. Fox originally alluded to a result which gives rise to this game in "A Primer for Chumps," which appeared in the *Saturday Evening Post*, November 21, 1959, and discussed the idea further in private correspondence arising from that article in a note entitled "A Curiosity in the Binomial Expansion—and a Lesson in Logic." I am indebted to Clayton Rawson and John Scarne for alerting me to Fox's paper.

Exact solution

These results, and others like them suggest that the nearest even integer to $1/(1-2p)$ is the optimal value of N, unless $1/(1-2p)$ is an odd integer and that then both neighboring integers are optimal. To assist in the proof of this conjecture, we let P_{2n} be the probability that $P(X \geq n+1)$ in a game of $2n$ trials:

$$P_{2n} = \sum_{x=n+1}^{2n} \binom{2n}{x} p^x q^{2n-x}.$$

In a game of $2n+2$ trials $P(X \geq n+2)$ is

$$P_{2n+2} = \sum_{x=n+2}^{2n+2} \binom{2n+2}{x} p^x q^{2n+2-x}.$$

A game composed of $2n+2$ trials can be regarded as having been created by adding two trials to a game of $2n$ trials. Unless player A has won either n or $n+1$ times in the $2n$ game, his status as a winner or loser cannot differ in the $2n+2$ game from that in the $2n$ game.

Except for these two possibilities, P_{2n+2} would be identical with P_{2n}. These exceptions are (1) that having $n+1$ successes in the first $2n$ trials, A loses the next two, thus reducing his probability of winning in the $2n+2$ game by

$$q^2 \binom{2n}{n+1} p^{n+1} q^{n-1};$$

or (2) that having won n trials in the $2n$ game he wins the next two, increasing his probability by

$$p^2 \binom{2n}{n} p^n q^n.$$

If $N=2n$ is the optimal value, then both $P_{N-2} \leq P_N$ and $P_N \geq P_{N+2}$ must hold. The results of the previous paragraph imply that these inequalities are equivalent to

$$1) \quad \begin{aligned} q^2 \binom{2n-2}{n} p^n q^{n-2} &\leq p^2 \binom{2n-2}{n-1} p^{n-1} q^{n-1}; \\ q^2 \binom{2n}{n+1} p^{n+1} q^{n-1} &\geq p^2 \binom{2n}{n} p^n q^n \end{aligned}$$

or, after some simplifications (we exclude the trivial case $p=0$),

$$2) \quad (n-1)q \leq np; \; nq \geq (n+1)p.$$

These inequalities yield, after a little algebra, the condition

$$3) \quad \frac{1}{1-2p} - 1 \leq 2n \leq \frac{1}{1-2p} + 1.$$

Thus unless $1/(1-2p)$ is an odd integer, N is uniquely determined as the nearest even integer to $1/(1-2p)$. When $1/(1-2p)$ is an odd integer, both adjacent even integers give the same optimal probability. And we can incidentally prove that when $1/(1-2p) = 2n+1$, $P_{2n} = P_{2n+2}$.

Generalization

I. R. Savage suggested to the author that the game be generalized. Again with the probability $p(<\frac{1}{2})$ of winning a single point, suppose that to win the game of N trials, one must have r more points than one's opponent. Again we want to find the value of N (unrestricted) that maximizes the probability of winning. The normal approximation suggests that the optimum value of $N \approx (r-1)/(1-2p)$. Suitably interpreted, this approximation is satisfactory as an exact solution. The methods used above in the exact solution for $r=1$ and N an even integer can be extended. It turns out after some algebra that for odd values of r, the optimum value of N is the positive odd integer nearest $(r-1)/(1-2p)$, while for r an even integer, the optimum value of N is the positive even integer nearest $(r-1)/(1-2p)$.

The distinction between odd and even values of r can matter substantially. For example, let $r=3$, $p=0.4$, then direct application of the approximate formula gives $N=10$, with probability 0.05476 of winning by at least 3 points, which for an even value of N means by at least 4 points. On the other hand for $N=9$ or 11, the probability of winning by at least 3 points (a win by exactly 3 points is now achievable) is 0.09935. Furthermore, the value $N=10$ is not the best *even* value of N to choose, because then winning by 3 implies winning by 4, and the optimum N for $r=4$ is $N=14$ (or 16), with probability of winning 0.05832.

Trends in the education of secondary-school mathematics teachers[1]

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What mathematics courses have been required of prospective teachers of secondary-school mathematics? Have changes in curricula for prospective mathematics teachers occurred since 1920?

ONE OF THE TOPICS of primary concern to mathematics educators and mathematicians in recent months has been the preparation of secondary-school mathematics teachers. This concern is evidenced in two recent reports of important groups of mathematicians, scientists, and educators. The 1960 report of a joint commission of the AACTE and AAAS on general problems of improving instruction in mathematics and science gave specific attention to the education of mathematics teachers.[2] In January of 1961, the Panel on Teacher Training of the Committee on the Undergraduate Program in Mathematics of the Mathematical Association of America issued in pamphlet form its recommendations for the education of mathematics teachers.[3] These recommendations, dealing with five levels of instruction, were also printed in this journal.[4] Individual authors have also discussed this topic in a variety of journals.

It is not the purpose of the present article to discuss or compare the specific proposals that are currently being made, nor does the author intend to offer suggestions of his own. The fact that important committees are making reports which they hope will be influential raises the question of how influential such reports have been in the past. Any attempt to answer this question necessarily involves a study of trends in the education of mathematics teachers and the possible relationships of any discovered trends to the recommendations of committees and individual mathematics educators. Such a study was carried out by the author and dealt specifically with the education of secondary-school mathematics teachers in the period from 1920 to 1958.[5] It is the purpose of the present article to summarize some of the findings of that study.

The study was undertaken at a time when the current ferment in mathematics education at all levels was already well under way, the Commission on Mathematics of the College Entrance Examination Board having issued preliminary re-

[1] Based in part on a talk given at the Teacher Education Section of the Summer Meeting of the NCTM, Salt Lake City, August 24, 1960.

[2] Joint Commission on the Education of Teachers of Science and Mathematics of the American Association for the Advancement of Science and the American Association of Colleges for Teacher Education, *Improving Science and Mathematics Programs in American Schools* (Washington: AAAS and AACTE, 1960). 41 pp.

[3] Mathematical Association of America, *Recommendations for the Training of Teachers of Mathematics* (Buffalo: MAA, 1961). 17 pp.

[4] Mathematical Association of America, "Recommendations of the Mathematical Association of America for the Training of Teachers of Mathematics," THE MATHEMATICS TEACHER, LIII (December, 1960), 632–38.

[5] John A. Schumaker, "Trends in the Education of Mathematics Teachers." (Ph.D. thesis, New York University, 1959. Library of Congress Microfilm 60–1112, copyright 1960 by John A. Schumaker.) 310 pp.

ports. There was considerable concern with, and uncertainty about, the mathematical content of the teacher-preparation program. It seemed to the author that any knowledge of factors influencing curriculum change which might result from historical investigation might be valuable in the determination of future practice.

The purposes of the investigation were (1) to determine the nature of curricula for the education of senior high school mathematics teachers in selected teacher-education institutions in the United States in the academic year 1920–21, (2) to determine the changes which occurred in these curricula over the period from 1920–21 to 1957–58 with emphasis on trends in the mathematical content of the curricula (the professional content of the curricula will not be discussed in the present article), (3) to determine differences in trends among teachers colleges, state universities, and other teacher-education institutions, (4) to attempt to ascertain reasons for the changes in curricula with emphasis on the possible influence of the recommendations of committees and of leaders in mathematics education, and (5) to suggest implications and recommendations for future practices.

For the purpose of the investigation a "teachers college" was defined as a degree-granting institution which had the preparation of teachers as its *primary* stated purpose during some portion or all of the period covered by the study: practically every one of these institutions had one of the words "teachers" and "normal" in its official name at some time during that period.

The term "state university" was defined to mean that institution which *originally* centered attention on the liberal arts and related professional schools as contrasted to the land-grant college which centered attention on agriculture and engineering. Liberal arts colleges were included among "other teacher-education institutions." These definitions were necessitated by the many name changes, especially of teachers colleges and land-grant colleges, that have occurred in recent years.

The basic working list for selection of the institutions included in the study was the membership list of the American Association of Colleges for Teacher Education as revised in March of 1957. This list included 355 four-year, degree-granting and accredited institutions whose membership in the Association was some indication of the seriousness of their intent in the teacher-education field. From this list, 41 schools were eliminated when a study of their 1957–58 catalogues revealed that they had no curricula for the education of senior high school mathematics teachers.

A questionnaire was sent to the remaining 314 institutions and those that graduated the largest numbers of qualified senior high school mathematics teachers in 1957 were selected for inclusion in the final study. It was found that the mean number of mathematics teachers prepared by teachers colleges was about double that for either of the other two types of institution. Wherever possible, one school of each of the three types was selected from each of the 47 states or geographical regions having members in the AACTE. Additional teachers colleges preparing large numbers of mathematics teachers were included to make a total of 140 schools in the final study.

The principal sources of data on course offerings and requirements were the catalogues of the selected institutions. One check on the accuracy of catalogue information in general was provided by asking the questionnaire respondents to verify data taken from the 1957–58 catalogues of their institutions; relatively few corrections were made. Catalogue data on undergraduate course offerings and requirements for prospective senior high school mathematics teachers were secured for the academic years 1920–21, 1928–29, 1936–37, 1943–44, 1950–51, and 1957–58. One of the factors in the selection of these particular years was the attempt to give

an optimum chance for the reports of certain important committees to show their influence.

A word about the situation prior to 1920 seems in order at this point. At that time the undergraduate mathematics required of prospective teachers seldom went beyond calculus in many of the teacher-education institutions. Attention was given to the professionalization of the mathematical subject matter, especially in the teachers colleges. Various proposals for improved teacher-education programs were generally in agreement on the inclusion of college algebra, trigonometry, analytic geometry, differential and integral calculus, history of mathematics, some attention to the foundations of algebra and geometry and to the applications of mathematics, a methods course in mathematics, and observation and practice teaching as essential components of these programs.

Of the 140 selected institutions, some 85 had teacher-education curricula in mathematics by 1920–21, as revealed by their catalogues for that year. It was found that elementary mathematics through calculus was offered by almost all of the schools. The most common advanced courses were differential equations, theory of equations, history of mathematics, advanced calculus, solid analytic geometry, and elementary projective geometry. The last-mentioned course was offered mainly in the state universities which, as one would expect, had a much more extensive offering of advanced courses at that time than did either of the other types of school. The mathematics major of 1920–21 who planned to teach had to meet a median minimum requirement of twenty-four semester hours of mathematics, including the usual freshman courses and a full year of calculus, plus other courses which were most likely to include solid geometry, differential equations, and theory of equations. The median minimum requirement for a mathematics minor preparing to teach was only 12 semester-hours of mathematics. The median minimum requirement in professional courses was 21 semester-hours.

The college mathematics offerings of all the selected teacher-education institutions with curricula for senior high school mathematics teachers were tabulated for each of the selected years. The results of these tabulations are presented in Table 1. This table shows the per cent of the institutions offering each course in each of the years, and it thus presents a picture of the changes that occurred in the five intervals that were included in the period of the study.

The usual freshman and sophomore courses are listed in the order in which they were generally studied; the other courses are listed in descending order of the per cent of institutions offering them in 1957–58, except that a second course in a given subject is usually listed immediately after the first course in that subject. In the tabulation of this data, catalogue descriptions were used to insure that courses tabulated under the same title were similar in content, although they might have had different catalogue titles.

It should be noted that in the last few years covered by the study, the offerings in analytic geometry and calculus were combined in a number of mathematics departments. There was no attempt to tabulate these courses in a separate category, but each was tallied under the appropriate individual titles as revealed by its description.

Table 1 reveals that work in college algebra, plane trigonometry, analytic geometry, and differential and integral calculus was offered by almost all of the institutions throughout the period covered by the study. The first three of these courses were integrated into unified freshman mathematics in about one-fourth of the schools by 1928–29, but that course was not more widely offered in any later year. The general education course in mathematics (often required of students not majoring in mathematics or physical science) first appeared in a substantial number of schools

TABLE 1

PER CENT OF TEACHER-EDUCATION INSTITUTIONS OFFERING MATHEMATICS COURSES IN SELECTED YEARS

COURSE	1920–1921	1928–1929	1936–1937	1943–1944	1950–1951	1957–1958
Solid Geometry	68	58	52	62	59	47
College Algebra	93	83	87	90	92	87
Advanced College Algebra	34	31	31	28	21	18
Plane Trigonometry	98	88	92	96	94	89
Analytic Geometry	94	91	89	93	94	87
Advanced Analytic Geometry	34	48	33	32	26	19
Unified Freshman Mathematics	8	25	21	20	19	23
Differential Calculus	98	98	99	99	100	100
Integral Calculus	92	92	98	99	100	100
Intermediate Calculus	9	4	6	13	20	24
Differential Equations	64	64	74	78	84	90
College Geometry	16	40	59	66	72	79
Advanced College Geometry	4	6	10	9	6	9
Theory of Equations	47	57	68	70	82	79
Advanced Theory of Equations	9	7	4	4	6	6
Mathematics of Finance	32	54	60	60	69	73
Elementary Statistics	24	41	53	62	65	72
Advanced Calculus	40	42	49	53	64	68
General Education Course	0	4	10	35	34	57
History of Mathematics	44	45	48	50	50	52
Solid Analytic Geometry	40	43	45	46	44	40
Modern Algebra	7	7	6	8	16	39
Vector Analysis	14	15	16	20	34	38
Elementary Projective Geometry	29	31	40	37	38	35
Theory of Numbers	11	10	11	17	31	34
Advanced Statistics	4	7	12	15	26	29
Fundamental Concepts of Algebra	6	6	11	11	19	29
Fundamental Concepts of Geometry	8	6	8	8	13	28
Arithmetic	16	17	26	30	35	26
Probability	11	8	13	14	17	26
Complex Variable	13	15	13	10	19	24
Numerical Analysis	0	1	1	4	9	24
Spherical Trigonometry	27	17	14	47	34	23
Slide Rule	2	1	2	5	16	21
Surveying	22	21	14	19	24	21
Higher Algebra	1	5	10	13	21	20
Theoretical Mechanics	31	26	25	20	19	19
Determinants and Matrices	4	3	2	4	5	18
Teachers' Course	22	17	16	16	11	18
Applied Mathematics	1	4	7	14	16	16
Partial Differential Equations	7	4	6	4	12	16
Real Variable	15	16	7	8	11	16
Higher Analysis	1	0	3	4	5	16
Analytic Projective Geometry	6	5	6	6	6	12
Differential Geometry	9	7	3	6	10	11
Elementary Topology	0	0	1	2	4	11
Finite Differences	2	4	5	5	12	11
Computer Programming	0	0	0	0	2	10
Higher Geometry	1	4	6	7	7	10
Navigation	0	1	1	23	9	10
Insurance	11	3	7	8	9	8
Non-Euclidean Geometry	1	2	3	4	8	8
Fourier Series	4	3	3	2	6	7
Quality Control	0	0	0	0	7	7
Actuarial Theory	5	4	5	5	6	6
Foundations of Mathematics	0	1	3	5	9	6

Note: The per cents in this table are based on 85 institutions for 1920–21, 114 for 1928–29, 126 for 1936–37, 133 for 1943–44, and 140 for 1950–51 and 1957–58.

during the middle interval of the study and was offered in a majority of them by 1957–58. Other relatively elementary courses which came to be offered widely were elementary statistics and mathematics of finance.

Table 1 shows that differential equations was the most widely offered advanced course throughout the 38-year period. College geometry was offered by only a few schools in 1920–21 but increased steadily in popularity, and in 1957–58 it was tied with theory of equations for second place among the advanced courses. The per cents of institutions offering theory of equations and advanced calculus increased steadily, and each was half again as large in 1957–58 as in 1920–21. There were no pronounced trends in the proportions of institutions offering history of mathematics and solid analytic geometry. The only other courses offered by more than 30 per cent of the schools in 1957–58 were modern algebra, vector analysis, theory of numbers, and elementary projective geometry; only the last-mentioned of these failed to show a substantial increase in listing subsequent to 1943–44.

It is interesting to note that three courses which were revived to a considerable extent during the war period, as reflected by the data for 1943–44, and later declined in relative frequency of listing, are solid geometry, spherical trigonometry, and navigation. New courses, such as computer programming and elementary topology, appeared in several schools during the last fifteen years of the period. A tendency throughout the period was for new courses to be added to offerings without any other courses being dropped.

The median minimum number of semester-hours of mathematics required of prospective teachers taking majors increased from 24 to 27 near the end of the period, and at the same time the corresponding requirement for a minor increased from 16 to 18 semester-hours. These changes, and also the number of hours required in professional courses, are shown in Table 2.

Not all of the colleges listed requirements for a major in terms of specific courses in the various years of the study. The per cent of those that did make such specific listings was tabulated for each course for each of the years. The result of this tabulation is given in Table 3. If two types of mathematics major existed in a school, only the one with the fewer requirements was tabulated.

Table 3 shows that the principal change in the list of courses beyond calculus likely to be required of a major was the addition of college geometry which headed the list in 1957–58; theory of equations, differential equations, and solid geometry remained high on the list throughout the period. The per cents of schools requiring advanced calculus and solid analytic geometry decreased considerably. There was agreement only on the freshman course and calculus; no other course was specifically required by more than one-third of the institutions in any of the selected academic years. It might be noted that the author compared his findings with those of any available status studies that were made during the period covered by

TABLE 2

MEDIAN MINIMUM NUMBER OF SEMESTER-HOURS REQUIRED OF PROSPECTIVE MATHEMATICS TEACHERS IN SELECTED YEARS

YEAR	SEMESTER-HOURS OF MATHEMATICS FOR MAJOR	SEMESTER-HOURS OF MATHEMATICS FOR MINOR	SEMESTER-HOURS IN PROFESSIONAL COURSES
1920–21	24	12	21
1928–29	24	16	22
1936–37	24	16	21
1943–44	24	16	22
1950–51	25	16	24
1957–58	27	18	24

TABLE 3

PER CENT OF INSTITUTIONS REQUIRING MATHEMATICS COURSES OF PROSPECTIVE TEACHERS MAJORING IN MATHEMATICS IN SELECTED YEARS

COURSE	1920–1921	1928–1929	1936–1937	1943–1944	1950–1951	1957–1958
Solid Geometry	25	18	21	19	18	17
College Algebra	78	82	84	88	85	82
Advanced College Algebra	16	6	14	11	9	7
Plane Trigonometry	69	86	73	84	88	82
Analytic Geometry	91	84	80	89	87	85
Advanced Analytic Geometry	13	25	15	9	11	5
Unified Freshman Mathematics	3	12	16	9	10	16
Differential Calculus	88	94	90	97	98	100
Integral Calculus	81	92	88	91	97	98
Intermediate Calculus	3	0	4	10	15	9
College Geometry	3	20	23	15	25	32
Theory of Equations	19	22	15	17	22	25
Differential Equations	22	20	16	17	22	18
History of Mathematics	6	16	12	9	13	12
Advanced Calculus	16	18	10	9	12	11
Fundamental Concepts of Algebra	3	2	0	3	4	11
Elementary Statistics	0	2	11	11	9	10
Mathematics of Finance	0	2	6	4	2	8
Teachers' Course	9	4	9	8	4	7
Fundamental Concepts of Geometry	3	0	0	4	2	6
General Education Course	0	0	0	5	2	5
Solid Analytic Geometry	16	20	11	13	12	5
Elementary Projective Geometry	6	2	1	0	2	1
Spherical Trigonometry	3	2	2	7	6	1
Arithmetic	6	8	6	3	0	0

Note: The per cents in this table are based on 32 institutions for 1920–21, 49 for 1928–29, 81 for 1936–37, 103 for 1943–44, 121 for 1950–51, and 133 for 1957–58.

the study; no serious discrepancies were found.

Tabulations corresponding to those of Tables 1, 2, and 3 were made separately for teachers colleges, state universities, and other teacher-education institutions. The offerings within each year as well as the changes from year to year were compared. These tabulations have not been included in the present article inasmuch as none of the results were particularly startling. The state universities offered more courses throughout the period than did either of the other types of school. Courses that the universities offered with considerably greater relative frequency were unified freshman mathematics, elementary statistics, advanced calculus, modern algebra, and fundamental concepts courses in algebra and geometry.

Throughout the period covered by the study, differential equations was the most widely offered advanced course in both the state universities and other teacher-education institutions; but college geometry took the lead among teachers colleges by the middle of the period and was found in 88 per cent of those schools in the 1957–58 academic year. Theory of equations was consistently offered with greater relative frequency than was college geometry in the state universities and other schools; but by 1957–58 each course had come to be offered in more than two-thirds of the schools of all three types.

Courses in surveying, navigation, and arithmetic were found almost exclusively in the teachers colleges, and these institutions shared the professionalized subject matter or teachers' course with the state universities.

In general, the teachers colleges gave fewer advanced courses than schools of either of the other types. There was no

general agreement among the schools of any type on the courses that a student majoring in mathematics should study after calculus; no course was required in as many as 40 per cent of the schools of any type in any academic year. College geometry and theory of equations were two of the three advanced courses required with greatest relative frequency in each of the types of school in 1957–58; the other course was history of mathematics in the teachers colleges, fundamental concepts of algebra in the state universities, and differential equations in the other schools. In earlier years differential equations had been among the leaders in all three types of school.

The semester-hour requirements for majors and minors were nearly the same for all types of school during the entire period. The professional requirement was greatest in teachers colleges, but the difference was slight.

The study gave special attention to the professionalization of subject matter in content courses in mathematics. The findings will not be discussed in the present article except to state that the questionnaire results showed that 1957–58 practice in this area was essentially the same as that found in two status studies made some 20 years earlier. In a related area, the most common audio-visual aid found to be in use in 1957–58 was colored chalk.

As background for the discussion of the influence of specific factors on changes in curricula for the education of mathematics teachers, the author made a survey of significant historical developments in general philosophy and curricular change in the senior high school and in teacher-education institutions. Specific attention was given to the evolution of the mathematics curriculum of the high school.

The reader is probably familiar with the fact that by 1920 the traditional college-preparatory sequence found in most high schools included advanced algebra, solid geometry, and trigonometry and that this underwent little change until the recent period of ferment. A significant development in secondary-school mathematics during the period covered by the study was, of course, the advent of so-called general mathematics as an alternative to elementary algebra in the ninth grade. The two-track program is commonly found in many high schools today.

The author also studied the evolution of teacher-certification requirements as shown by data obtained for the six selected years. The median minimum number of semester-hours of mathematics required for certification to teach senior high school mathematics was found to be 15 during the last two decades of the period covered by the study. This is only slightly less than the requirement for a minor in mathematics in the selected institutions as shown in Table 2.

Statements found in the literature with regard to the influence of various factors on changes in curricula for the education of senior high school mathematics teachers reflected diverse opinions held by the writers. At one extreme was the opinion that such curricula were stimulated largely within departments and that outside agencies had little effect on them. At the other extreme was the opinion that a particular committee report was the source for the organization of most training courses for mathematics teachers. Most writers held a view somewhere between these two extremes. Respondents to the questionnaire had few opinions to offer concerning such possible influence. To investigate such possible influence, the author turned his attention to recommendations for teacher education that were made during the period covered by the study.

Several widely recognized groups of mathematics educators made recommendations for teacher-education curricula during this period. The proposals of the National Committee on Mathematical Requirements (1923), the Commission on Training and Utilization of Advanced Students of Mathematics (1934), the

Joint Commission on the Place of Mathematics in the Secondary Schools (1940), the Commission on Post-War Plans (1945), and the Madison Symposium on Teacher Education in Mathematics (1952) were carefully studied and compared.

The recommended mathematics content of the program was found to vary little among the proposals. That content is typified by the list of courses recommended as essential by the Commission on Training. The courses in the list for mathematics as a minor field of senior high school teaching were college algebra, trigonometry, analytic geometry, differential and integral calculus, and college geometry; for a major field, the additional courses were advanced algebra such as the theory of equations, mathematics of investment, a first course in mathematical statistics, and directed reading or a formal course in the history and fundamental concepts of mathematics. These courses total approximately 18 semester-hours for a minor and 30 semester-hours for a major in mathematics, somewhat above ascertained minimum practice as shown in Table 2.

Proposals for teacher-education curricula that were made by several leaders in mathematics education and by various researchers were compared with the commission reports, as were the results of certain studies of high school teachers' opinions on this subject. All of these proposals and opinions were found to bear a striking similarity to each other which might have indicated that a suitable ideal program had been attained; however, the continued appearance of proposals indicated that such was not the case.

The repeated recommendations of college geometry and theory of equations by the various groups, starting with the National Committee in 1923, may be reflected in the increased per cents of the selected teacher-education institutions that offered and required these courses, the change being more marked for the former course which was the only course recommended by all of the groups as a required course.

There was, however, found to be an absence of specific evidence of the influence of committee reports and other factors on changes in teacher-education curricula in mathematics in the reports of institutional evolution that the author investigated. But this absence itself provides some evidence that these factors had little direct effect on curricula.

Starting with the report of the Commission on Training, the proposed programs seem to represent practice already in effect in a majority of the institutions insofar as course offerings are concerned. That the commission recommendations and other factors had little effect on the requiring of specific courses is evidenced by the fact that in only one instance was any course beyond calculus required of prospective teachers by more than 25 per cent of the selected institutions in any academic year for which data were obtained.

Many of the recommendations for the inclusion of courses such as statistics, mathematics of finance, and history of mathematics in teacher-education curricula in mathematics were based on the changed character of the secondary school.

In many high schools, general and consumer mathematics became a substantial part of the mathematics offerings during the period covered by the study and many of the pupils in the elementary algebra and geometry courses were interested in applications and appreciations of these subjects in everyday life. Other mathematics courses included in proposals for teacher-education programs and often found in programs in practice were also directly related to courses taught in the secondary school. This was especially true of the college geometry course which was included in virtually all of the proposals and was actually required in more programs than was any other course beyond calculus. The relationship of calculus to the usual secondary-school mathematics was often less evident, especially to the high school teachers themselves, but such

a relationship nevertheless existed.

In general, it appears that secondary-school mathematics curricula had a definite effect on teacher-education curricula in mathematics. The usefulness of courses as background for teaching secondary-school mathematics was taken into consideration in proposals and programs. The relatively stable nature of senior high school college-preparatory mathematics over the 38-year period was reflected in the similarity of the various proposed programs for teacher education. As previously noted, the principal effect of teacher certification requirements was to establish a floor for the number of semester-hours needed for a minor in mathematics in the teacher-education institutions.

In the current ferment, the first group to make significant proposals for changes in teacher-education curricula in mathematics was the Commission on Mathematics. The Commission was concerned with the implementation of its program for secondary-school mathematics. These proposals were first released in pamphlet form in 1957 and were later incorporated in the final report of the Commission. The Commission called for the elimination of theory of equations from teacher-education programs and stated that it considered non-Euclidean geometries, topology, and the elements of differential geometry as more appropriate than the usual college geometry course.[6] We have agreed not to compare this and more recent reports in the present article. We are concerned, rather, with the possibility of these proposals exerting a real influence on curricula.

An encouraging note for those in sympathy with the proposals of the Commission is offered by the 1958 questionnaire findings of the author. Of 111 respondents to a question dealing with plans for curriculum revision, 64 indicated that some revision was planned. About 40 per cent of the planned revisions could be considered as in the direction of "modernization" of curricula; in several instances new courses in geometry and modern algebra were to be introduced. One respondent indicated that his school intended to offer courses in "Modrun algebra and modurn geometry [*sic*]." The trend to modernization was more pronounced in the teachers colleges than in the other types of institution. Many other revisions not related to this trend were, however, listed by individual respondents to the questionnaire.

However, the previously cited findings of the study indicate that the implementation of any currently desired changes in the curricula for mathematics teachers is very likely to be a long and slow process.

The Commission on Mathematics properly recognized the role of the secondary-school mathematics curriculum as a principal determining factor of teacher-education curricula. The eagerness of many teachers in service in recent years to enroll in special courses, such as those of National Science Foundation Institutes, is quite significant in this respect. Many of these teachers are motivated by the desire to introduce topics of "modern mathematics" into the curricula of the schools in which they teach or by the necessity of doing it because the decision has already been made in their school. However, it remains to be seen how pronounced the trend toward modernization of senior high school college-preparatory mathematics will become. Also, there is no indication whether general mathematics and other non-college-preparatory courses in the secondary schools will be significantly affected by this movement.

The findings of the study indicate that, in the absence of such definite changes in secondary-school mathematics, there is little likelihood that significant changes in the curricula for mathematics-teacher education will occur soon.

Substantial agreement among various

[6] Commission on Mathematics of the College Entrance Examination Board, *Program for College Preparatory Mathematics* (New York: The Board, 1959). Pp. 55, 56.

committees and individual leaders in mathematics education as to the desirability of certain proposed changes should have a beneficial effect in the accomplishment of those changes. Past evidence indicates that any changes will come very slowly and will be evidenced first by the addition of new courses to the lists of offerings in a number of teacher-education institutions; these schools will be quite reluctant to drop any existing courses from the lists. The inclusion of a new course among requirements for prospective senior high school mathematics teachers will come very slowly. No such course beyond calculus is likely to be required in a majority of the institutions unless some factor not present in the 38-year period covered by the study enters the picture.

It will be very interesting to see what effect the present period of ferment will have on teacher education in mathematics. There will be need for follow-up studies of the status of curricula. The author hopes to conduct such a study for the 1964–65 academic year using the 140 selected institutions of the study reported here. There is also a great need for an investigation of the effectiveness of present teacher-education programs in meeting the needs of teachers in modern, comprehensive high schools and for experimental studies of the effectiveness of various approaches to courses within curricula, as well as of the complete mathematics content of different curricula.

Contemporary Mathematics to be repeated on Continental Classroom in 1961-1962

The National Broadcasting Company announced on June 28 that last season's "Continental Classroom" course in Contemporary Mathematics will be repeated on color-tape recordings from 6:00 to 6:30 A.M. local time, beginning September 25, 1961.

Contemporary Mathematics, the new course on "Continental Classroom" in 1960–1961, is taught by John L. Kelley, Professor of Mathematics at the University of California, Berkeley, and Frederick Mosteller, Professor of Mathematical Statistics and Chairman of the Department of Statistics at Harvard University. Professor Kelley teaches Modern Algebra during the first semester of Contemporary Mathematics; Professor Mosteller teaches Probability and Statistics during the second.

The new course on "Continental Classroom" during 1961–1962 will be a course in American Government. Dr. Peter H. Odegard, one of the nation's most distinguished political scientists, now Professor of Political Science at the University of California, Berkeley, will teach the two-semester, college-level course, which will deal with the structure and function of the U.S. Government. The course in American Government will be televised in color and carried by approximately one hundred seventy stations in every part of the country, Monday through Friday, from 6:30 to 7:00 A.M. local time, starting September 25.

The Conference Board of the Mathematical Sciences is one of the sponsors of Contemporary Mathematics; the others are Learning Resources Institute and the National Broadcasting Company.

"Portable demonstration laboratories are enriching science instruction in the upper elementary and junior high school grades."—*From "Schools in Our Democracy," Office of Education, U.S. Department of Health, Education, and Welfare.*

HOBLADIC Mk I Mod 1

GEORGE W. HORTON, *Illinois College, Jacksonville, Illinois.*

Here is a refinement of the "do-it-yourself" computer known as HOBLADIC.

A FURTHER SOPHISTICATION of HOrton's BLAckboard DIgital Computer[1], increasing its sensibilities and the complexity of its capabilities, is made by supplying it with a "moving accumulator." By a mere statement, then, HOBLADIC MkI Mod 1 is hereby invented. It will have two registers, named *Upper* and *Lower*, actually side by side, each being the extension of the other, all movable to the right or left as many places as instructed. The two registers may in some instances be considered as separate entities although operations begun in one may carry over into the other. In other cases the registers may be considered as one double-sized accumulator. They shift as a unit.

In actual construction, a sliding slab of blackboard is arranged to move laterally across the top of the array used in HOBLADIC MkI. A prominent indicator is used to mark the division between the upper register at the left and the lower register at the right.

Figure 1*a* shows the slide in its normal position. In this example, 1,023 is in the upper and 76,001,529 is in the lower. This might also be interpreted to represent 102,376,001,529 which had resulted from, say, an addition or a multiplication. Shifting to the left 2 places gives Figure 1*b*, in which 102,376 is in the upper and 152,900 is in the lower, it being noted that 8 digits on either side of the divider are considered as being in the register. Now a shift to the right of 5 places produces the display in Figure 1*c*, with 1 in the upper and 2,376,001 in the lower.

[1] George W. Horton, "HOBLADIC," THE MATHEMATICS TEACHER, LIV (April, 1961), 212–16.

USE OF THE SLIDE

Since the slide may be moved either to the right or to the left by an instructed number of places, it may be used for (a) dividing or multiplying the number in the accumulator by the corresponding power of 10, or (b) for moving all or any part of a number from one accumulator to the other.

CODED INSTRUCTIONS

Several of the instructions applicable to MkI must be restated in the new framework, and several new instructions will be needed.

ADC—Accept Data Card. (Unchanged)
DLC—DeLiver Card. (Unchanged)
AUP—Add to UPper. (Augmentation of ADD)

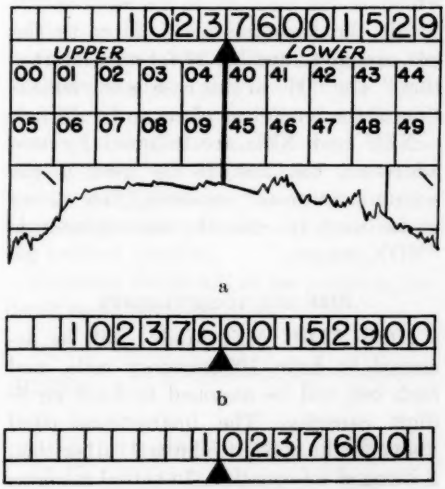

Figure 1

ALO—Add to LOwer. (Augmentation of ADD)
SUP—Subtract from UPper. (Augmentation of SUB)
SLO—Subtract from LOwer. (Augmentation of SUB)
MPY—MultiPlY number in Upper, product shows in Lower.
DIV—DIVide number in Upper, quotient shows in Lower.
SLT—Shift accumulator(s) to the LefT the indicated number of places.
SRT—Shift accumulator(s) to the RighT the indicated number of places.
EWL—Erase and Write in the Lower.
EWU—Erase and Write in the Upper.
CLU—CLear the Upper. (Read-in as CLU 00.)
CLO—Clear the LOwer. (Read-in as CLO 00.)
STU—STore the Upper accumulator.
STL—STore the Lower accumulator.
NZU—Test Upper for Non-Zero.
NZL—Test Lower for Non-Zero.

Particular attention must be called to the instructions SLT and SRT, the shifting instructions. A typical instruction would appear as

22 SLT 03 27

which is the instruction in number 22: "Shift the accumulator 3 places to the left, and go to number 27 for next instruction." The "03" in this case is the indication of the *number of places* to be shifted.

NZU and NZL are followed by two addresses, the first to be used if the accumulator tests "non-zero," the second to be used in case the accumulator is "NON-nonzero."

Size and acceptability

HOBLADIC MkI Mod 1 will be assumed to have 100 memory cells, and each cell will be assumed to have an 8-digit capacity. The instructions cited above might appear to have 9 rather than 8 items of information. In actual machine practice, however, an instruction such as MPY would be read-in as a 2-digit code number for MPY. This paper uses the approximately cognate 3-letter expression as it is more easily followed.

Data read-in will remain 00-04 and 05-09; answer read-out will remain 40-44 and 45-49.

The number and complexity of the problems that can be solved has been greatly increased by the modifications set out. It may be also noted that some problems that were solvable before can now be more simply resolved. Only two problems will be considered, one completely, the other in outline only.

Weighted averages

The computer in its present form may be conveniently used to compute weighted averages. One example might be:

> Find the yield, total cost, and cost per pound of a mixture made up on Formula 37 which calls for 11 pounds at 63¢ per pound, 15 pounds at 75¢ per pound, 18 pounds at 55¢ per pound, etc.

Data:

$$\frac{00}{37001163}, \frac{01}{00001575}, \frac{02}{00001855}, \text{ etc.} \cdots$$

It should be noted that the data in 00 contains three separate numbers, 37 being the number of the formula, 11 being the number of pounds of one of the substances used, 63 being the unit cost of this material. The program must first separate these individual numbers and then use them as the problem requires. In sequence:

ADC	00	
EWL	00	37,001,163 in lower
SLT	02	37 in upper; 00,116,300 in lower
STU	40	37 in number 40 for identification
CLU	00	Upper clear
SLT	04	11 in upper; 63,000,000 in lower
STU	39	11 in number 39 for future reference.....(a)
CLU	00	Upper clear

```
SLT  02   63 in upper; lower clear
MPY  39   11×63 in lower
STL  38   693 in number 38.....(b)
CLU  00\  Slide should be centered when
CLO  00/  both registers are clear.
ALO  01
SLT  06   15 in upper; 75,000,000 in lower
STU  37   15 in number 37.....(a)
CLU  00   Upper clear
SLT  02   75 in upper; lower clear
MPY  37   75×15 in lower
STL  36   1,175 in number 36.....(b)
CLU  00
CLO  00
ALO  02
SLT  06
STU  35   18 in number 35.....(a)
CLU  00
SLT  02
MPY  35
STL  34   990 in number 34.....(b)
```

The last 9 instructions constitute the routine for each component.

```
CLU  00
CLO  00
AUP  39
AUP  37
AUP  35   etc., collecting the (a) terms
STU  41   the yield, 44 in number 41
EWU  38
AUP  36
AUP  34   etc., collecting the (b) terms
STU  42(*) total cost, 2858, in number 42
DIV  41   average cost, 64, in lower
STL  43   average cost, 64, in number 43
DLC  40   card delivered, reading:
```

$$\frac{40}{37},\quad \frac{41}{44},\quad \frac{42}{2858},\quad \frac{43}{64},$$

or, "Formula number 37 will yield 44 lb. at a total cost of \$28.58, this being an average cost of 64¢ per pound. The 64¢ is an integral answer with no consideration to fractions of a cent. If, when the figure, 2858, was in the upper accumulator, at (*), a shift to the left of two places had been made before DIV 41, the lower accumulator would have shown 6497, or 64.97¢ per pound.

There is practically no limit to the applications and ramifications of this general process. It could be applied to the determination of average grades where courses are not of equal weight. Considerably embellished, it serves as a basis for the prediction of election results from meager early returns.

The decimal point

Consider the simple product,

$$P=0.236\times40.3\times0.033.$$

Here the problem is not only to obtain the product of the combinations of digits shown but to locate the decimal point. There are several methods of doing this; here is one that is readily understood.

Rewrite the product as

$$P=236\times10^{-3}\times403\times10^{-1}\times33\times10^{-3}$$

or yet as

$$P=236\times10^{9}\times403\times10^{11}\times33\times10^{9}\times10^{-3\times12}$$

establishing 10^{12} as the datum of decimal reference. The numbers would then be read-in as 09,000,236, 11,000,403, and 09,000,033, the first two digits in each case being the power of 10 in reference to 10^{12} as datum. Each number would be placed in turn in the lower, shifted 2 places to the left, the exponent picked out of the upper and stored, another shift to the left of 6 places putting the multiplicand in the upper where it is subject both to storage and to multiplication. A collection of 12's can also be made, one for each number in the original product.

Omitting the details of the program, the result could be read-out as

$\frac{40}{3138564}\cdots$, the product, $236\times403\times33$,

$\frac{41}{29}\cdots$, the sum, $9+11+9$, the sum of the exponents

$\frac{42}{36}\cdots$, 3×12, the cumulative datum figure.

Thus the answer would come from the computer as

$$3{,}138{,}564 \times 10^{29-36}\ 0.3138564.$$

It is to be noted that HOBLADIC avoids the concept of negative numbers. If it be further assumed that the computer can differentiate between positive and negative numbers, the scope of operation is increased. The manipulation of decimal points is one of the obvious areas of such an application.

Conclusion

A recent fresh look at the literature on commercial computers discloses that the fundamentals of programming are generally universal, not unique to any one of them. One of them handled decimal points almost move for move like the above program. The fundamentals of programming may be learned on and from any machine. The virtue in HOBLADIC lies in the fact that you are always able to *see* the numbers in process. If you learn to handle, and above all to *watch* the figures as they develop in HOBLADIC, the translation into terms of any commercial computer will not be difficult.

What's new?

BOOKS

College

Boolean Algebra and Its Applications, J. Eldon Whitesitt. Reading, Massachusetts: Addison-Wesley Publishing Company, Inc., 1961. Cloth, x+182 pp., $6.75.

Calculus and Analytic Geometry, Robert C. Fisher and Allen D. Ziebur. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1961. Cloth, xv+766 pp., trade edition, $12.65; text edition for classroom adoption, $9.50.

Elementary Algebra for College Students, Irving Drooyan and William Wooton. New York: John Wiley & Sons, Inc., 1961. Cloth, x+272 pp., $4.95.

Elements of Linear Algebra, Lowell J. Paige and J. Dean Swift. Boston: Ginn and Company, 1961. Cloth, xvi+348 pp., $7.25.

Finite Difference Equations, H. Levy and F. Lessman. New York: The Macmillan Company, 1961. Cloth, vii+278 pp., $5.50.

Linear Algebra, G. Hadley. Reading, Massachusetts: Addison-Wesley Publishing Company, 1961. Cloth, ix+290 pp., $6.75.

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Simplified Calculus, F. L. Westwater. New York: The Macmillan Company, 1961. Cloth, xv+160 pp., $3.50.

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High School

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Advanced High School Mathematics, Glen D. Vannatta, Walter H. Carnahan, with Harold P. Fawcett, consultant. Columbus, Ohio: Charles E. Merrill Books, Inc., 1961. Cloth, xvii+420 pp. List, $4.96; net, $3.72.

Algebra in Easy Steps with Modern Units, Edwin I. Stein. Princeton, New Jersey: D. Van Nostrand Company, Inc., 1961. Cloth, vi+302+74 pp., $3.96.

A Course in Geometry—Plane and Solid, Arthur W. Weeks and Jackson B. Adkins. Boston: Ginn and Company, 1961. Cloth, vii+552 pp., $4.40.

Mathematics for Junior High, Anne Peskin. Larchmont, New York: Youth Education Systems, Inc., 1960. Paper, 96 pp., $1.50.

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Senior Technical Mathematics, A. H. Heywood (editor). Toronto, Canada: The Macmillan Company of Canada, Ltd., 1960. Cloth, xii+558 pp., $4.50.

A new introduction to the ideas and methods of trigonometry

ROBERT J. THOMAS, *DePauw University, Greencastle, Indiana.*

Part of the effectiveness of this introduction lies in the mathematical system itself, and part of it lies in the method of presentation.

THERE ARE THREE USES for the mathematical system herein developed. It may be used briefly in a trigonometry course as an introduction to the basic ideas and methods. It may be used more extensively in a teacher training course to make the teachers-to-be think in the way that they should make their future students think. It may be used in an advanced class, even more extensively, to study the development of a mathematical system per se. It is in regard to the first of these that this system was originally developed and has been used by the author in college classes. It may be used soon in a high school teacher-training course, and it is being considered for use in an advanced seminar.

The need for a means of focusing the student's attention on the basic ideas and methods of trigonometry was felt by the author and others on the staff because of the poor trigonometry background of many college students. Many high school courses, it seems, have defined the trigonometry functions solely in terms of triangles and have almost exclusively concentrated on the solution of triangles. Whereas it was felt that the sine, *et al.*, should be thought of primarily as functions of paired numbers (x,y) plus a third number associated with and calculated from the pair. The facts that this pair is a point in the Cartesian plane, that the third number is the distance from the origin, that this distance forms an angle with the positive x axis, and that the functions are useful in solving triangles are all incidental, even though very useful.

College students are very likely to have been exposed to some of the terminology of trigonometry in high school, even if not to a short course. This makes it difficult to focus the student's attention on the new ideas using the old symbols and definitions. Even though new definitions are given, some students continue to think in terms of the old ones until it is too late to make the change-over. The names of the trigonometric functions are likely to be associated with triangles in the mind of the student, and the usual arguments (α, β, γ, θ, ϕ, etc.) are likely to be thought of as angles *of triangles*. The practical ends which trigonometry may serve, i.e., solution of triangles, may also tend to obscure the means.

The following introduction to the ideas and methods of trigonometry attempts to lead up to the functional approach while overcoming the above difficulties. Part of the effectiveness of this introduction lies in the mathematical system itself, and part of it lies in the method of presentation. Brief examination of some new, unheard of, and completely useless functions is used to introduce the ideas and operations of trigonometry, one at a time. Set theory is not explicitly mentioned in the following discussion, but can be mentioned and used simultaneously with this introduction to trigonometry. This is especially so since the beginning point in this presentation is the set of all two-member subsets of the set of all real numbers

(Cartesian product), and the notation used is the same as that usually used for the set of all rational numbers or the set of all complex numbers.

The form of this paper is primarily that of an outline for the college teacher, which means that there are places where details need to be filled in or elaborated upon and where further illustration or discussion needs to be supplied. Nonetheless, it should be readable and understandable, as is, by the better student. This is especially so since in many cases the rather precise manner of presentation to the student has been spelled out in keeping with the importance of the actual method of presentation of this system mentioned earlier. Very slight modifications of this paper will make it adaptable to the other persons and uses discussed above.

In connection with the presentation and elaboration of this system, two things are important. The first is to let the student explore any and *all* possibilities and to make the discoveries himself as much as possible, i.e., not to rush the presentation of the results, to be sure that the student at least understands the existence of a problem and what the problem is before giving him a solution. The second is to emphasize the distinction between results which are themselves arbitrary decisions and results which follow as consequences of previous arbitrary decisions, the latter being the more numerous. In connection with the first, for instance, the first sum of functions, which is one, should be developed; the student should then be given a chance to develop the others before the results are given him. The student, of course, should not be expected to learn and remember the results as he would have to do in trigonometry proper. This will enable him to focus his attention on the methods and ideas of development and relation.

Six functions

We start with any two arbitrary numbers. Since we cannot write both of them at the same time and/or in the same space, we must order them by writing one of them first, to the left of the other. The pair (2,3) would be one of the many possible examples. Since we cannot write down all possible examples, it will be useful to use the technique of algebra and let x stand for the first number and y stand for the second, so that in general we have a pair (x,y).

We now arbitrarily define a third number $z=x+y$. Continuing with our example, we have $z=5$. Our number z has no particular significance. It is a function of x and y. It is only one of many possible functions of x and y which we could have chosen, but it has the advantage of simplicity of both concept and calculation. Note that for every given (x,y) there is a unique value of z associated with them and which may be calculated from them. These are the conditions which are necessary for z to be called a function of x and y, usually written $z=z(x,y)$ or $z=f(x,y)$. Note also in this case that it is equally easy to find the x which goes with a given y and z or to find the y which goes with a given x and z.

From here on we will be concerned with the study of the six ratios which can be formed with these three numbers, taking them two at a time. These ratios have no particular significance, and there is no special reason for studying them except curiosity and practice in developing systematically whatever relations may exist among a set (group, collection); although if we wanted to do arithmetic with the three numbers x,y,z, these ratios express all the divisions involving two numbers which we may encounter. Of course, division by zero is excluded.

Now, for convenience, and only for convenience, we assign a name to each of the ratios, as follows:

$x/z=$ bad, $\quad z/x=$ dab,

$y/z=$ bid, $\quad z/y=$ dib,

$x/y=$ bud, $\quad y/x=$ dub.

Note that although the first ratio is called "bad," there is nothing worse about it than about any of the others. Errors are sometimes made in the reasoning processes because a value judgment is associated with a technical name like this. Also note, again for convenience, that the name of the reciprocal of any ratio is the name of the ratio spelled backwards.

Identities

Now, finding ourselves with six ratios, we will investigate the arithmetic of these ratios. Of course, the result of adding bad to bad is 2 bad. Leaving out this type, there are fifteen possible sums and fifteen possible differences. This is quite a few to investigate, but many can easily be ruled out as useless.

The question which may be in the mind of the student at this point is "Why?" Why should we throw away the results of some of these operations and keep others? The answer is mainly that mathematicians are men, and men are lazy. Even though we are investigating a completely useless system, we will still want to retain relations which simplify any calculations or algebra which may arise and to discard all others.

As an illustration, let us take the first two ratios in the left-hand column. We can add or subtract, so we try both and compare.

$$\text{bad}-\text{bid}=\frac{x}{z}-\frac{y}{z}=\frac{x-y}{z}.$$

And here we are stuck with an awkward expression, so why bother to remember it? If we should be so unfortunate as to run across such a subtraction, we can do the necessary arithmetic then (probably by computing bad and bid and then subtracting).

On the other hand,

$$\text{bad}+\text{bid}=\frac{x}{z}+\frac{y}{z}=\frac{x+y}{z}=\frac{z}{z}=1,$$

because of the basic definitions. Now this is worth remembering! If we should be so fortunate as to run across this combination, we would not have to do any computation. We would know that the result would be 1 regardless of the values of x, y, and z. The "regardless of" makes this expression useful, and also makes it, by definition, an identity.

Now let us try the same thing with dib and bud, adding first this time.

$$\text{dib}+\text{bud}=\frac{z}{y}+\frac{x}{y}=\frac{z+x}{y}.$$

Again we are left with a result that is not too appealing. However,

$$\text{dib}-\text{bud}=\frac{z}{y}-\frac{x}{y}=\frac{z-x}{y}=\frac{y}{y}=1$$

is a useful, labor-saving device, and also an identity.

The student can verify that dab$-$dub $=1$ and that the sum, dab$+$dub, is not useful.

It should be noted that we have used all of the combinations having common denominators. It should also be noted that if adding gave a useful result, subtracting did not, and vice versa. The question now arises as to whether this will always be true. (It will be seen that there is one rather strange exception.)

The subtraction of reciprocals is another type of identity that lets us replace certain ratios in certain combinations with other ratios.

$$\begin{aligned}\text{dab}-\text{bad}&=\frac{z}{x}-\frac{x}{z}=\frac{z^2-x^2}{xz}\\&=\frac{(z-x)(z+x)}{xz}=\frac{y(z+x)}{xz}\\&=\frac{yz+yx}{xz}=\frac{yz}{xz}+\frac{yx}{xz}\\&=\frac{y}{x}+\frac{y}{z}=\text{dub}+\text{bid}.\end{aligned}$$

The student should verify that:

$$\text{dib}-\text{bid}=\text{bud}+\text{bad},\quad \text{bud}-\text{dub}=\text{dib}-\text{dab}.$$

These do not seem to be as useful as they might be, i.e., it would be nice if we could replace a pair of ratios by just a single ratio. However, by transposing a term, we can do even better. For instance, dab − bad − dub = bid. We will return to this kind of consideration later, but first let us complete the list of identities.

The sums of the reciprocals give expressions that will not factor and will not simplify. (The student should try one of these.) Three other combinations appeared on the right-hand side of each of the last three identities, so we have now considered fifteen of the possible thirty additions and subtractions. If we reverse the sign of the right-hand side of these last identities, we obtain the strange expressions mentioned earlier.

$$\text{dib}+\text{dab}=\frac{z}{y}+\frac{z}{x}=\frac{zx+zy}{yx}=\frac{z(x+y)}{yx}$$

$$=\frac{zz}{yx}=\text{dib}\times\text{dab}.$$

The sum is equal to the product!

Since the identities have been appearing in groups of threes, the student may have anticipated that there are two more expressions of this type. This is indeed the case, and the student should verify that:

$$\text{dub}-\text{bid}=\text{dub}\times\text{bid},$$

$$\text{bud}-\text{bad}=\text{bud}\times\text{bad}.$$

This leaves six more possibilities. They are all of the type,

$$\text{bad}\pm\text{dib}=\frac{x}{z}\pm\frac{z}{y}=\frac{xy\pm z^2}{zy},$$

and so do not yield very useful identities.

Having now thoroughly looked at addition and subtraction and having found some relations between these operations and multiplication, the next step is to look at the other twelve pairings for multiplication. Three of these are in the form of a ratio times its reciprocal, the result of which is 1. Of the remaining nine, six are useful.

$$\text{bad}\times\text{dib}=\frac{x}{z}\times\frac{z}{y}=\frac{x}{y}=\text{bud},$$

$$\text{bad}\times\text{dub}=\text{bid},\quad \text{bid}\times\text{dab}=\text{dub},$$

$$\text{bud}\times\text{dab}=\text{dib},\quad \text{bid}\times\text{bud}=\text{bad},\quad \text{and}$$

$$\text{dib}\times\text{dub}=\text{dab}.$$

As our first general identity, let us look at bid×bad the hard way, and then the easy way.

$$\text{bid}\times\text{bad}=\frac{y}{z}\times\frac{x}{z}=\frac{y}{z}\times\frac{z-y}{z}=\frac{zy-y^2}{z^2}$$

$$=\frac{y}{z}\left(1-\frac{y}{z}\right)=\text{bid}\ (1-\text{bid}).$$

This is the hard way, using definitions, direct substitution, and a lot of arithmetic and algebra made necessary because we have not made use of one of the basic identities which we have already established, namely, bad + bid = 1. Using this, bid×bad = bid×(1 − bid) in just one step! If you are not lazy and not interested in saving work, time, energy, and paper, either way is good; otherwise, the latter is certainly preferable. You can just as easily get the alternate expression, bid×bad = (1 − bad)×bad. Which of these two we would use, that is whether we would want to express the result in terms of bid or bad, depends upon what we want to combine it with or to compare it with next.

Before moving on to the usual type of identity problem, let us look at two from the other side, that is, let us construct, rather than solve, the problem. We start with something simple, like 0 = 0. We can write this 1 − 1 = 0. If we replace either "1" with any of our simple identities we have a new identity, such as bad + bid − 1 = 0 or 1 − dib + bud = 0. We are usually presented with an expression like the latter two and asked to show that they really are identities, i.e., true for any value of the variables x,y,z. This we do by working backwards, ending with the obviously true statement 0 = 0.

If we were asked to show that bid ×(1 − bid) is identically equal to bad

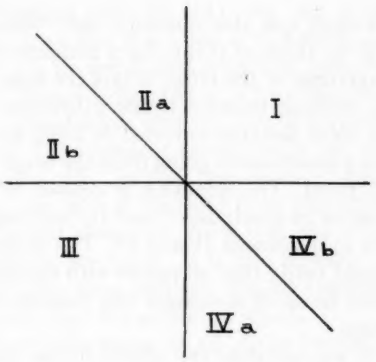


Figure 1

$\times(1-\text{bad})$, we could show (as we did above) that each of these is equal to bid$\times$bad. Thus an identity may be established by showing that each side is identically equal to some third quantity.

The student should make up some more identities and also establish that the following are identities:

$$-(\text{bad}\times\text{bud})+\text{bid}+\text{bud}=1,$$

$$\text{dab}-\text{dub}-\text{bid}=1-\text{dib}+\text{bud}+\text{bad}.$$

Signs

It needs to be emphasized that this section is not a necessary consequence of the previous, nor is it necessary for most of what follows. Up until now it has been emphasized that x and y were just a pair of numbers, and this is all that *need* be said of them. However, they can also, in addition, be thought of as the ordinary Cartesian co-ordinates. If we add this latter meaning, it is then of interest to note the signs of the various functions in the different quadrants. We should note, however, that the reciprocal of a ratio will always have the same sign as the ratio; so we need only consider three of the ratios, say bad, bid, and bud.

We will find it necessary to divide two of the quadrants in half, as indicated in the diagram in Figure 1. In quadrant I, x and y are both positive; and so z and all of the ratios will be positive. In quadrant III, x and y are both negative; so z will be negative. This again makes all the ratios positive. In quadrant II, y is positive and x is negative; so z can be either positive or negative, depending upon whether x or y has the larger absolute value. Semiquadrant IIa is the region where $y>x$, so z is positive; semi-quadrant IIb is just the reverse. The bud will be negative in both of these regions, but bad is negative in IIa and positive in IIb while bid is just the reverse. Quadrant IV is similarly divided, with z negative in IVa.

The student should verify the signs of the functions in regions IVa and IVb; these are given, along with the other results from above, in the following table.

I	Everything positive.
IIa	$x<0$, $y>0$, $z>0$; bad<0, bid>0, bud<0.
IIb	$x<0$, $y>0$, $z<0$; bad>0, bid<0, bud<0.
III	x, y, and z negative; bad, bid, and bud positive.
IVa	$x>0$, $y<0$, $z<0$; bad<0, bid>0, bud<0.
IVb	$x>0$, $y<0$, $z>0$; bad>0, bid<0, bud<0.

Note that the functions have the same signs in IVa as in IIa and in IVb as in IIb (and in III as in I).

Note that on the slanting line dividing the a regions from the b regions, x and y have the same absolute values and opposite signs; thus z is zero and bad and bid are undefined because of the zero denominator. Dab and dib are zero and bud (also dub) is negative (as it is on either side of the line). The student should investigate the conditions on the x axis and on the y axis.

Argument

So far we have just discussed the functions of one pair of numbers at a time. That is, for instance, the bad of a pair plus the bid of the same pair. (Most of this section will be presented with the original viewpoint that x and y are just a pair of numbers. However, the added concept of

the Cartesian graph of a point can be used when desired and may help in visualization. It must be emphasized again that this added concept is not *needed* and that the first viewpoint is quite sufficient.) We will soon want to compare and add ratios of different number pairs; and, to do this, we must have some way of distinguishing the bad of one pair from the bad of another pair.

To make this distinction, we can use any property associated with the pair that we wish as a label to identify it. Thus we are now faced with a major arbitrary decision. We could just give each pair a number or a letter and talk about bad (2) or bid (a), that is, the bad of pair 2 or the bid of pair a. Or we could use x^2, so that bud (4) would mean the bud of the pair with x equal to 2 (so that $x^2=4$). Or we could use $2(x-y)$. We could use any of an infinite number of other quantities. We must just decide what we *want* to use and then stick to this convention. This identification quantity, or label, is called the "argument."

The case using x^2 mentioned above could be troublesome for two reasons. First, both pairs under discussion might have $x=2$. We would still be unable to distinguish between them without devising an additional method. This objection would not be very troublesome if it were not combined with the second: that the functions of the two pairs would not be the same, that is, bud (4) $\neq$ bud (4). Since this is at least disconcerting, if not confusing, we will want to pick a method of distinguishing between pairs that does not lead to this situation.

Of the many, many ways of designating a pair, only a few will satisfy this restriction. One such method is given by

$$\frac{|y|}{|x|+|y|}$$

to which is prefixed the sign of the product xy. The sign can be taken care of automatically by writing this as

$$\frac{xy}{|x|\cdot(|x|+|y|)}.$$

We shall call this quantity the "offset."

If we think of points for a moment, the magnitude of the offset is just the ratio of the distance traveled in the y direction to the total distance traveled in both the x and y directions in going from the origin to the point. The quantity is *defined* to be positive in quadrants I and III and negative in quadrants II and IV. The student should verify that all points with the same offset lie along a straight line through the origin.

If we calculate the offsets of the pairs (1,2), (−3,1), (−2,3), and (3,6), we find that they are 2/3, −1/4, −3/5, and 2/3. Two of these pairs have the same offset, so our first objection is not overcome; but, since the second objection is overcome, this does not matter. That is, the ratios associated with points with the same offset will be equal.

To show that if the offset of two pairs is the same, then the values of all the corresponding ratios will be the same, we consider two cases. If x and y have the same sign, then the offset is just $y/(x+y)$ or y/z; but this is just bid. So two pairs of this type with the same offset will have the same value for bid. Since bad = 1 − bid, they will have the same value for bad. The reciprocals dab and dib will obviously have the same values for each pair. Since bud = −1 + dib, bud will have the same value for each pair (and therefore so will dub).

If x and y have opposite signs, the offset is given by $y/x-y$. So if the offset of two pairs is the same, say pair a and pair b, we have

$$\text{(A)}\qquad \frac{y_a}{x_a-y_a}=\frac{y_b}{x_b-y_b}.$$

We also have the obvious identity

$$\text{(B)}\qquad \frac{x_a-y_a}{x_a-y_a}=\frac{x_b-y_b}{x_b-y_b}.$$

Adding (A) to (B) we get

$$\text{(C)}\qquad \frac{x_a}{x_a-y_a}=\frac{x_b}{x_b-y_b}.$$

Dividing (C) by (A), we get $x_a/y_a = x_b/y_b$, or bud of a = bud of b. Adding (A) and (C) we get z_a and z_b for the numerators. Then dividing (A) or (C) by this relation we establish that the bids or bads are equal. (The student should carry out the details.) Or, having established equal buds, we can use the three basic identities, as we did above, to show that the other functions are equal. It is also possible to graph the pairs and use similar triangles to show that the ratios are equal.

From this point on, all of the functions will be written with an argument, like bad .5 or bud a, meaning the bad function of the pairs with an offset of .5 or the bud function of pairs having an offset of a, where a is a variable.

Note that this addition of an argument to distinguish which pair we are discussing does not invalidate anything that has gone before. We really should, though, for consistency, go back and rewrite everything by adding an argument; but since this is a little impractical we will be satisfied with just always using one from here on.

Two other things should be noted. One is that bad a is not equal to bad $-a$, even in absolute value. For instance, bad .3 = .7, while bad $-.3 = 7/4$. This will be discussed further in the section on inverses.

The other thing to be noted is that the absolute value of the offset varies between 0 and 1, inclusive; the signed value varies between -1 and $+1$, inclusive, with -1 being the same as $+1$! This function reverses sign on going through 1 the same way as functions usually do on going through 0; that is, there is no more distinction between $+1$ and -1 than there is between $+0$ and -0! The offset of the x axis is ± 0; the offset of the y axis is ± 1; and the offset of the line dividing the a regions from the b regions is $-\frac{1}{2}$.

Graphs of the six ratios, plotted as functions of the offset, are given in Figure 2.

In conclusion it should be pointed out that we are rather fortunate in this case to be able to compute so readily the functions from the offset. This is possible since they were both defined in terms of x,y,z. Sometimes, as in trigonometry, the argument chosen in a system like this is such that the functions cannot be computed directly from it. In such cases, tables are compiled so that, given the argument, the functions can be looked up, or vice versa.

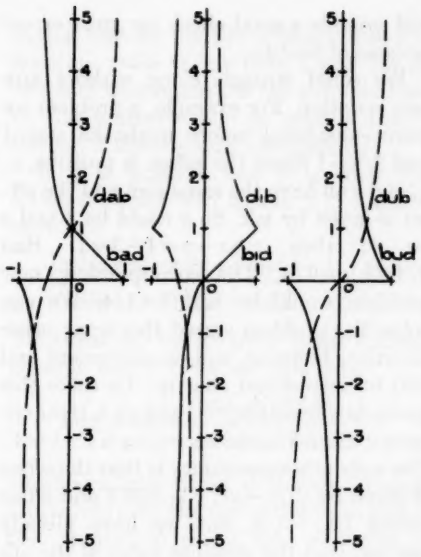


Figure 2

Inverse functions

In the very beginning we defined z as a function of x and y, so that given x and y we could determine z. We then noticed that given z and either x or y we could proceed backwards and determine the unknown x or y. We now want to consider the six functions in a similar way. Up to now we have been given the offset and from that determined the value of one or more of the six functions. We now want to reverse the procedure; given the value of the function, we want to determine the offset.

One method, of course, would be to use the graphs of the functions. Since this is not very accurate we will want to devise algebraic methods. The graphs, however,

will provide a good check for gross errors in cases of doubt.

We could struggle along without any new notation. For example, a problem we have considered before might be stated bad $3/4 = ?$ Since the offset is positive, x, y, and z all have the same sign and the offset is given by y/z. So y could be 3 and z be 4; then $x = z - y = 4 - 3 = 1$. Bad $3/4 = x/z = 1/4$. The corresponding new problem would be bad $? = 1/4$. We can solve the problem stated this way; other notation, however, will be convenient and will be introduced shortly. To solve this problem we can take $x = 1$ and $z = 4$. If the offset is y/z, it is $3/4$ (since $y = z - x = 4 - 1 = 3$). The only other possibility is that the offset is given by $y/(x-y)$; y is still 3 and offset would be $-3/2$, but we have already noticed that the absolute value of the offset cannot exceed 1 (so this latter answer is impossible and the previous answer has to be the right one). Note that we could have picked different initial values, but that the answer would have been the same because only ratios are involved. For instance, in the last problem we could have taken x to be -1 and z to be -4; then y would have been -3 and the offset $-3/(-4) = 3/4$, as before. Or $x = 2$ and $z = 8$ would have given a bad of $1/4$, y of 6, and offset of $6/8 = 3/4$.

Now let us turn to the new notation. First, a "?" is seldom used, so let us replace it by some usual variable notation, say u. The first problem above would now be $u =$ bad $3/4$. Note that we have the variable isolated on one side of the equation. The second problem would be bad $u = 1/4$. Note that the variable to be solved for is not isolated and cannot be isolated. It is certainly convenient to be able to isolate it, and new notation will allow this to be done. Two terms often used for this type of procedure are "anti" and "arc"; that is, if $y = f(x)$, then $x =$ anti-$f(y)$ or arc-$f(y)$. However, it will be a little clearer in this system if we use "off" for this purpose. We can then restate the second problem, $u =$ offbad $1/4$, to be read "u is the offset whose bad is $1/4$." One more example is $u =$ offbud $1/2$. (u is the offset whose bud is $1/2$.) Since bud is x/y, we can take x as 1 and y as 2 giving us z is 3 and u is $2/3$. (Picking any other values for x and y that give a bud of $1/2$ will give this same answer.)

If the student wishes, he may compare this procedure with his work with logs. $u = \log 100 = 2$. Conversely, $2 = \log u$ gives $u = 100$; but it is much more convenient to write $u =$ antilog $2 = 100$ (u is the number whose log is 2).

Relation to trigonometry

It is the method of beginning, developing, and studying the above system that is important and has wide applicability, not the details or results. Trigonometry is begun in exactly the same way, with seven definitions, six of which are ratios. The number pair x,y is used, but instead of z we have r which is defined to be $\sqrt{x^2+y^2}$ (which means it is always positive). Six ratios are then formed in the same way and named for convenience. Two of the ratios in trigonometry are the same as two of the ratios in the system we have just developed, but have different names; bud is cotangent and dub is tangent. Identities are developed and used in exactly the same way as are signs of the various functions in various quadrants. The ratios are plotted as functions of the argument, but the argument used is an ordinary angle instead of the offset. Inverses are treated in the same way, with "arc" being the notation used ($u =$ arctangent 1 is read u is the angle whose tangent is 1, which also means tangent $u = 1$).

The thing which is really important is to note how all the developments depend upon and go back to the arbitrary definitions. If the definitions are kept well in mind and used, the rest follows easily, clearly, and unambiguously.

A final note

There is a rule of thumb in mathematics that says that if a change of any kind is

made which causes a simplification somewhere, a complication arises elsewhere. This is the case with this system discussed above. While this far it has been simpler than or comparable to trigonometry (z is more simply defined than r, identities are first-power rather than second-power relationships, the argument is limited rather than unlimited, etc.), the addition relationships are much more complicated. An introduction to this topic and some of the elementary results are given below for the more advanced or interested person.

The problem is to find relations such as bad $(a+b)$ in terms of bad a and bad b. In neither trigonometry nor the present system is this as simple as a straightforward addition of the two functions. In trigonometry there are general formulas which apply regardless of which quadrants the angle is in; in the present system different formulas have to be developed to cover the various possible combinations of regions in which a and b may be.

First of all, for the present system the sum has to be reduced modulo 2 and written as a number between -1 and $+1$ inclusive (since the offset cannot be outside this range—this also fits with the graphical meaning). Thus $.4+.7=-.9$; $.4+(-.7)=-.3$; $.5+.9=-.6$. We can summarize by saying that we use the ordinary sum if its absolute value is less than 1 and use the ordinary sum minus 2 if the absolute value of the ordinary sum is greater than 1 (either, if the absolute value equals 1).

If a is positive and b is such that $a+b>0$ (which also means ≤ 1), then we have:

$$\text{bad}(a+b)=\text{bad}\,a+\text{bad}\,b-1, \quad \text{and}$$

$$\text{bid}(a+b)=\text{bid}\,a+\text{bid}\,b.$$

Note that the last of these is the simplest possible result. Contrast this with

$$\text{bud}(a+b)=\frac{(\text{bud}\,a)(\text{bud}\,b)-1}{2+\text{bud}\,a+\text{bud}\,b},$$

which is a little tricky to derive, although it involves only the use of the basic identities. Bud$(a+b)$ could also be found by dividing bad$(a+b)$ by bid$(a+b)$.

If a and b are positive and their sum is negative we have:

$$\text{bad}(a+b)=\frac{\text{bad}\,a+\text{bad}\,b-1}{2(\text{bad}\,a+\text{bad}\,b)-1};$$

$$\text{bid}(a+b)=\frac{2-\text{bid}\,a-\text{bid}\,b}{3-2(\text{bid}\,a+\text{bid}\,b)}.$$

The formula for bud$(a+b)$ was not derived; it can be done, but it is quite involved algebraically.

Have you read?

HEIST, PAUL, *et al.*, "Personality and Scholarship," *Science*, February 10, 1961, pp. 362–367.

As a teacher of mathematics, you are often called upon to counsel students in respect to institutions of higher education they may wish to attend. This is a report on a study to determine if the intellectual environment of a college affected the quality of a student's work. National Merit scholars were chosen as subjects, and matched groups were compared, with the variable being a high-productive college as opposed to a low-productive college. The colleges were cataloged according to criteria of Knapp and Greenbaum. Measures were made of personality factors, theoretical, aesthetic, religious, and vocational interests.

In some areas of interest, there seems to be no difference. But you will be interested to note that students in high-productive institutions seem to have less conservatism, more objectivity, and more freedom for learning. They are theoretical and abstract while less inclined to technical aspects and business. In summary, it seems those attending high productivity schools have traits more closely related to serious intellectual pursuits. This article will make you stop and think when offering advice.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

*The game of Euclid**

A. HENRY ALBAUGH, *Hillsdale College, Hillsdale, Michigan.*

Here's an idea for the mathematics club party.

ALTHOUGH MATHEMATICS is a fascinating subject for many high school students, there are many more who do not share this enthusiasm. These students lack interest, or (let's face it) are just plain bored with the entire subject. Geometry is no exception and, if the subject is not taught carefully, the potentially good student is also apt to lose interest.

This was the problem that confronted the author when he began teaching high school mathematics. Other teachers had aids in their classes to induce the student to learn and to make the subject a bit more interesting; e.g., teaching history with the aid of maps. But what could be used to teach geometry?

One day it was noticed that several students were playing gin rummy—and with a great amount of spirit. "If only they were as interested in geometry as they obviously are in cards," I thought. "If only the game of rummy could be combined with the principles of geometry. . . . It might be unorthodox, but it's worth a try."

Two years later, when a sample game and set of rules had been completed, the product was offered to my class: Euclid, named after the man who invented geometry as we know it today. The game was an immediate success with the students, who would even venture into the mathematics room during their free periods to "play a hand or two."

It is easy to learn how to play Euclid, and anyone with a few weeks' knowledge of geometry should be able to do quite well. In basic content, this game resembles gin rummy. But there is one difference. There are two decks for Euclid: a players' deck of 100 cards, each card containing a statement or a reason, and a problem deck of 35 cards, each card having a geometric figure with all given parts marked and a statement to prove. (One possible logical solution to the proof is given on the back of the card.)

Each player tries to rid his hand of cards by placing on the table in front of him a statement or reason pertaining to the problem card, but these cards must be played in logical order; i.e., each statement card must always be followed by its corresponding reason card before the next logical statement card is played. The cards are drawn and discarded in much the same manner as in rummy.

The discarding plays as big a part as the actual laying down of the cards in that the player must look ahead in the game to see whether a certain card will be needed eventually. If he slips up and discards a card that is valuable, his opponent will then have the chance to pick it up and use it himself—if he is awake and notices the discard. It is the drawing and discarding that do the most toward the actual learning process involved in the game.

The game is officially over when a person has played his entire hand, regardless of whether the problem has been completely solved. The winner is the player with the most cards laid out before him.

The game is played with 100 playing cards and 35 problem cards. The rules for playing the game are similar to the rules of such popular card games as rummy or 500 rum. Nevertheless, the students actu-

* Although this game was developed by Professor Albaugh, this article was written by one of his students, Tom Hinks.

ally solve geometry problems while playing the game, and it is possible for the better student to win more often. There is, however, enough element of chance so that almost anyone will win his share of the time. If one were to walk unannounced into a room while the games were in progress, one would never dream that these students were solving geometry problems. The smiles and enthusiasm are certainly rather unusual. These students seem to be enjoying themselves too much.

Usually four or five students will sit at a table and try to outplay each other. Partners also play against each other.

Now how does this game go?

1 There are two decks: one a players' deck of 100 cards and the other a problem deck of 35 cards, as shown in Figure 1.

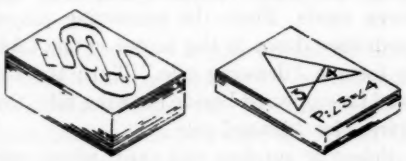
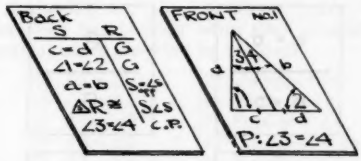


Figure 1

2 The players each receive five to seven cards from the players' deck and one card is chosen from the problem deck. This problem is then solved by the group in the same manner that any geometry problem is solved. Namely, *a statement is made* and the corresponding reason is given in a logical pattern. In this case, the student first puts down a card that has the correct statement. As long as he has cards that will continue the chain of logic he may play. (That is, if he has the correct reasons.) Then, the next player continues this chain of logic. (He places the reason card if the first student cannot.)

The solution to the problem is written on the back of the card (Fig. 2a), but the students are encouraged not to use this information unless a dispute arises.



a b

Figure 2

The front of the problem card (Fig. 2b) has a problem with *all the given parts marked* and a statement to be proved.

The cards needed to solve the problem are either in the balance of the deck or are, in a random fashion, held by the four or five players. The students play their cards only in a logical statement-reason order. When a student's turn comes, he draws a player card from the deck (or discard pile) and then proceeds to lay down *one or more* of his cards that fit in the chain of logic. He then discards one card on a discard pile and the next player continues in the same way.

The cards displayed in Figure 3 are the completed proof for the problem card shown in Figure 2. The left-hand column contains statements, the right-hand column shows reasons. The cards are read as follows:

Prove: Angle 3 = angle 4

S. (Statement): $c = d$.
R. (Reason): Given
S.: Angle 1 = angle 2.
R.: Given
S.: Therefore, $a = b$.
R.: Sides opposite equal angles of an isosceles triangle are equal.
S.: Therefore the triangles are congruent.
R.: Side, angle, side equal.
S.: Therefore angle 3 = angle 4.
R.: Corresponding parts of congruent triangles are equal.

Two of the ten cards needed to complete the proof are also shown in Figure 4.

Most of the symbols on the cards are self-explanatory.

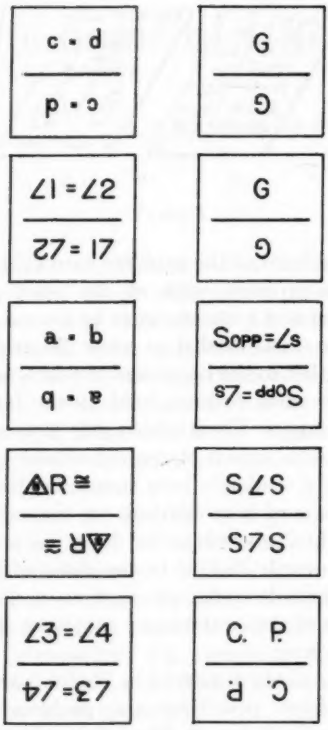


Figure 3

(a) G means given.
(b) S opp = ∠s means sides opposite equal angles in an isosceles triangle are equal.
(c) △R≅means triangles are congruent.
(d) S∠S means side angle side equals side angle side.
(e) C. P. means corresponding parts.

Since the statements and reasons can be used over and over in other problems, we need only 98 cards to solve 35 increasingly difficult problems.

We assume that the student has an acquaintance with most of the theorems learned in, say, the first 12–18 weeks of geometry.

I feel that this game is not only useful in making some classes more interesting and unusual, but that it has definite commercial possibilities in the educational field. I suspect that the students might

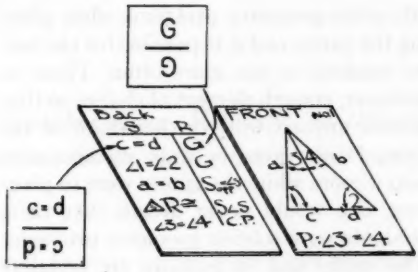


Figure 4

even involve their parents in an occasional game of Euclid.

Euclid rules

There are two decks of cards: (1) player cards, (2) problem cards.

Shuffle the player deck and deal one at a time face down until each player has seven cards. Place the remaining player cards face down in the center of the table to form a "drawing pile." Turn the top card face upward beside drawing pile; this starts the "discard pile."

Select at random one card, which contains a theorem to be proved, from the problem deck. Given parts are so designated, and are colored red.

Play is begun by player at left of dealer, who draws a card from the top of either pile.

Each player tries to get rid of his cards by laying down, face up before him, a statement or reason pertaining to the problem card. His cards must be played in a logical order; each statement card must always be followed by its corresponding reason card before another statement card may be played. (One possible logical order is on the back of each problem card.) When the player has played as many cards as he can *or wishes*, he then discards, and the next player proceeds in like manner, with everyone working on one group problem, rather than individual ones.

When all the cards in the drawing pile have been used, the discard pile should be

turned over face down to form a new drawing pile.

If one problem has been solved and no player has cleared his hand of all cards, another problem is begun and play continues.

The game is ended only when a player clears his hand, calling out "YUKE!"

The card marked "Yuke!" is a wild card, and can take the place of any other card, statement or reason.

Each player finds his score by counting his cards on the table.

Note: These rules are subject to change, since there are several different ways to play gin rummy.

Following are the 100 player cards. The figures in parentheses show the number of identical cards included in one deck.

(5) $\angle 1 = \angle 2$	(2) $\angle s$ opp. $= S$
(2) $\angle 1 = \angle 1$	(2) S opp. $= \angle s$
(2) $\angle 1 = \angle 3$	(2) Sup. of $= \angle s$
(2) $\angle 2 = \angle 3$	(2) Comp. of $= \angle s$
(2) $\angle 2 = \angle 4$	(2) $V \angle s R =$
(4) $\angle 3 = \angle 4$	(3) A.I. $\angle s$
(2) $\angle 5 = \angle 6$	(2) $= s - = s$
(2) $\angle A = \angle B$	(2) $= s + = s$
(5) $a = b$	(2) $= s \div = s$
(4) $c = d$	(2) $= s \times = s$
(3) $e = e$	(2) ls R//
(2) $u = k$	(2) rt. $\triangle$
(2) $x = y$	(5) $\triangle$ R $\cong$
(2) $\angle S \angle$	(4) Q's = to = Q's
(3) $S \angle S$	(9) G.
(2) S S S	(3) I.
(2) $h \angle$	(5) C.P.
(2) h l	(1) YUKE!

National Science Foundation summer fellowships for secondary-school teachers of science and mathematics

As one means of improving the teaching of science and mathematics in American secondary schools, the National Science Foundation plans to award on March 15, 1962 several hundred summer fellowships for secondary school teachers of science and mathematics. Those who receive fellowships must pursue a program of *graduate-level* work in the subject matter of science or mathematics. The fellowships will be awarded to support individually planned programs of study in the mathematical, physical, and biological sciences at a level that is acceptable by the fellowship institution toward an advanced degree in any of these subject matter disciplines.

The closing date for receipt of applications is January 2, 1962. Information and application forms will be sent upon request. Address requests to Secondary School Fellowships, American Association for the Advancement of Science, 1515 Massachusetts Avenue, N.W., Washington 5, D.C.

A teacher may apply for fellowship support for one, two, or three summers. Each applicant may choose the college or university at which he wishes to study—there is no list of college and universities selected for this purpose. The fellowship program is completely separate from the NSF summer institute program. It is recommended that a program of study for the period of the fellowship be developed with the assistance of a staff member at the proposed fellowship institution. The program of study is a very important part of the application. Although some colleges and universities will not have announced their course offerings for the summer of 1962 by the deadline date for applications (January 2) the proposed program of study in the application should list specific courses from which the courses for study in the summer of 1962 will be chosen.

Prior teaching experience in a secondary school is required of all applicants, and no exceptions will be made to the requirements outlined in the announcement. Those without the requisite teaching experience and those who must complete undergraduate prerequisites before undertaking a program of graduate-level work should not apply for a fellowship under this program.

• EXPERIMENTAL PROGRAMS

Edited by Eugene D. Nichols, Florida State University, Tallahassee, Florida

Notre Dame's 1960 summer program for gifted high school children

by Arnold E. Ross, University of Notre Dame, Notre Dame, Indiana

Motivation and history

Every teacher is fascinated by a gifted youngster and feels particularly responsible for such a youngster. Every professional, sooner or later, faces the problem of transmitting his own dedication and mastery to another generation. Our own involvement began to transcend these natural concerns as soon as we found ourselves in the role of mentors to the secondary-school teachers in our special programs for the teachers of mathematics.

In trying to decide the scope of the curriculum for the teacher, we were compelled to re-examine our concept of what mathematical ideas are appropriate for the very young and accessible to them. The least that we could do was to make sure that our curriculum for the teacher should provide the mastery of the requisite ideas and skills.

In searching for the suitable form of instruction, we thought of teaching as an art of developing in the very young curiosity, a sense of adventure, and a sensitivity of perception, as well as the mastery of the basic skills. As he practices his art, the teacher, naturally, should keep in mind the development of the facility in the use of language appropriate to each stage of the intellectual development of the pupil. We came to realize rather quickly that the teacher's mastery of the art of teaching, as we conceived it, does not come easily and that this mastery is certainly not an obvious consequence of some degree of familiarity with factual information. It appeared that the teacher's performance reflected his whole intellectual outlook.

It became quite clear that to produce the desired effect upon his charges, the teacher himself had to become committed to the habits of observation, conjecture, and critical reappraisal, as well as to the habit of a precise and concise use of language in his own thinking. In addition, he had to become aware of the limitations of language as the mechanism for the transmission of ideas, skills, and attitudes to the very young and inexperienced.

What seemed clear to us had to be made convincingly clear to our teacher-students. Some direct participation in work with gifted pupils was certainly indicated, for nothing is as convincing as a successful demonstration.

The transition from an academic interest in the able youngsters to a very personal involvement occurred in almost imperceptible stages. The way led from discussions with the teachers and the parents, to counselling the students, to working with about a half-dozen gifted boys in the summer of 1957, and to establishing in the academic year 1957–58 a Sunday class dubbed "Sunday School" by the youngsters and open to all able children in the environs of Notre Dame. Both the Sunday school and the summer term programs have continually grown in size and scope.

The principal object of our present re-

port is the summer program for "high-ability secondary-school students" sponsored at Notre Dame by the National Science Foundation in the summer of 1960.

Our summer group consisted of boys and girls who came from 15 states. These children represented 18 public, 18 parochial and 3 private schools. There were among them 8 seniors, 29 juniors, 14 sophomores, and 3 freshmen.

Their avowed interests ranged from engineering through the physical sciences to mathematics. Biological sciences were also represented, though these were decidedly in the minority (2 out of 54).

Whenever the selection of participants uses some measure of their potential, inasmuch as it can be judged through the quality of their initiative, intensity of their preoccupation, and the degree of their perseverance, one is bound to net a group representing a wide spectrum of interest.

At the outset, therefore, the planner is confronted with the dilemma as to what purpose should be served by a mathematics program for a collection of young individuals who have in common only eagerness, curiosity, an unbounded (and hitherto undirected) supply of vitality, and, possibly, an ultimate destiny in science.

One may decide upon a selection of some mathematical skills in the hope that these may prove useful at a later date. One may select a (reasonably simple) range of applications and treat jointly the mathematical tools and the ideas of the related science. One may, on the other hand, take advantage of the fact that with a proper selection of material one can remain within mathematics and yet exhibit a whole gamut of vital dilemmas which confront any scientist in any field at some time or another. This last is particularly true today when "theorizing" has become such an important tool in almost every field of human activity.

Even though we did not intend to neglect entirely any of the above approaches, we did subordinate everything to the aim of providing our charges with a rich experience in scientific thinking while still remaining within mathematics.

As the principal vehicle to carry us toward our objective we selected number theory. Since most of the youngsters have a great deal of experience with arithmetic, the subject is accessible to them. If taught properly, number theory provides an unusually fine opportunity for the development of the student's powers of observation, his intellectual curiosity, and his capacity for intellectual adventure. As an important by-product, the student acquires facility in the effective, precise, and concise use of language.

Number theory and the algebra of classes (studied concurrently with number theory) arise out of very basic, one may even say primitive, physical experience. It is extremely instructive for the student to participate in the natural development of a subject into a deductive science by letting him share in the continual interplay of observation, experiment, conjecture, and counterexample or proof. It is also very important for the student to realize that the asking of good questions is as important for the growth of an individual or, for that matter, for a science as the answering of questions already posed.

Academic organization

The number theory course involved four lectures a week given to the whole body of participants. In addition to attending the lectures, each student took part in three problem seminars a week. These seminars allowed for discussion in small groups. Further opportunity for informal out-of-class discussions was provided by the presence in the dormitories of very able and dedicated junior counsellors. This group of counsellors consisted of a small number of our own ablest mathematics majors, one gifted boy from Harvard, and one very able girl from the College of Mount St. Joseph. The use of gifted

undergraduates as resident counsellors living near the participants in their dormitory quarters was first tried in the summer of 1959 and proved to be quite successful.

The participants in our summer program varied greatly in the quality of their mathematical experience. The more mature among them were allowed to attend a specially designed course in higher algebra which reinforced and expanded the objectives of the number theory course. It may be mentioned at this point that, in its turn, number theory, as a birthplace of modern algebra, provided the background for a fuller ultimate appreciation of the ideas of abstract algebra.

A very small number of students who took part in our program in 1959 and 1960 were ready for the more technical courses in algebra, in geometry and in analysis. Such courses are a part of our summer program for the teachers* of mathematics. Each summer we have had with us a small group of high school youngsters with particularly high qualifications. They have always received very kind and sympathetic treatment from the senior members of our staff in residence at the time.

A project seminar and a project counselling service were organized under the direction of a member of our staff whose interests cover both mathematics and physics. This was done in recognition that projects are the most prevalent form of student participation activity in the high school. Although a significant proportion of our young people did come for counselling, we feel that as a rule projects in their traditional form do not provide the best outlet for a student's energies, particularly in a short and intensive summer session.

The student reaction

The students seemed enthralled even if somewhat intimidated by their first serious contact with ideas and by the large measure of responsibility which they had to assume for their own progress. Competition among the participants was very keen, even if friendly, and contributed much to the maintenance of high standards. The students learned to use the library for supplementary reading and for gathering information related to topics of special interest to them. The availability of a first-rate library appeared to be such a new experience for most of them that in their excitement they withdrew from our library in the first few days of the summer term a host of books on every conceivable subject. It took about ten days for the participants to satisfy their library hunger, to settle down, and to apply themselves to the systematic part of the program.

Most of the students wrote to us expressing the hope that they might return to Notre Dame in the summer of 1961. We regretted profoundly that we did not have sufficient funds at our disposal to bring even a half of our 1960 participants back.

The staff

Our senior staff consisted of: Professor Milko Jeglic (St. Mary's College, Notre Dame, Indiana): problem seminars; Professor Cecil Mast (University of Notre Dame): project seminar, project counselling, problem seminars; Professor Arnold Ross (University of Notre Dame): number theory lectures; Professor Hans Zassenhaus (University of Notre Dame): algebra lectures; Mr. Frank Long, Director of the Chemistry, Mathematics, and Physics Library: library counselling.

Our counsellors were: Catherine Herr, junior, College of Mount St. Joseph, Mount St. Joseph, Ohio; James Burton, sophomore, Harvard University, Cambridge, Massachusetts; and Robert Burckel, senior, Frank Owens, junior, Richard Jensen, junior, and James Wirth, senior, all of the University of Notre Dame, Notre Dame, Indiana.

* We shall be very glad to send upon request a descriptive pamphlet of 15 pages entitled "To Teach How to Teach How to Do" as long as our supply lasts.

Selection

Each applicant is sponsored by his mathematics teacher who is already familiar with the objectives of our program or who can be made aware of these objectives through correspondence. We have found that in most cases the interested teacher can help us decide whether the youthful prospect is ready for what we have to offer. The final decision to accept or reject a candidate is arrived at by a committee of our teaching staff on the basis of discussion with the teacher-sponsor and of other available evidence of the applicant's achievement and interests such as an autobiographical application letter, a special questionnaire, and his academic record.

Food for thought

We must say that we do not find it easy to single out those qualities of character and achievement which could be considered as an indication of promise. We try not to be too rigid in our selective process for fear of overlooking gifted nonconformists.

In the light of our experience with the Project Seminar, we are considering replacing this seminar with the study of some topics in geometry with which the student would not ordinarily come into contact. We hope that such a study in an informal seminar can utilize more efficiently the energy left over from preoccupation with the basic part of the summer program.

Remaining within the boundaries imposed by their immaturity, we have tried to bring our young people into contact with the very best scientific thinking of the day. Natural and inevitable limitations in scope which would allow us to set accessible, even if very ambitious, objectives, caused us to stop short of the vital concerns characteristic of experimental science. At present we do not have the resources to deal with experimental science, and we are disturbed because we cannot direct our young people to anyone who would attempt to do for them in experimental physical science what we have attempted to do for them in mathematics and in abstract thinking generally.

We feel that it would be highly desirable for everyone in our summer group to continue throughout the year some activity which maintains the quality of challenge presented by his summer experience. Unfortunately, for most of our participants this is impossible at present.

Working with the very gifted is a very exacting task. One is amply repaid for the effort, however, by the sight of the deep excitement and the keen pleasure shown by these young people as they meet and grasp new ideas and become aware of new horizons.

Have you read?

Fehr, Howard, "What Mathematics Is Taught in European Schools," *The American Mathematical Monthly*, October, 1960, pp. 797–802.

Much has been written on the comparison of European and American schools, but I think this short article points out clearly some crucial comparisons. For example, 30% of the 12-year-olds enter college preparatory schools, 15% continue to the equivalent of grades 9 and 10, and only 8% continue to grade 12. Modern materials such as set notations, probability, trigonometry of real numbers, axiomatic development, and non-Euclidean geometries are not touched. Calculus is taught, as is cosmography, transformation in geometry, theory of numbers, and conics.

You will be interested to note that the U.S. is carrying out a dynamic program in mathematics while Europe at present is relatively static.—Philip Peak, *Indiana University, Bloomington, Indiana.*

• HISTORICALLY SPEAKING,—

Edited by Howard Eves, University of Maine, Orono, Maine

The problem of Apollonius[1]

by N. A. Court, University of Oklahoma, Norman, Oklahoma

Historical introduction

In the history of Greek mathematics, the first two places belong to Euclid and Archimedes. The third place is usually reserved for Apollonius. The fame of the "Great Geometer," as he was called by his contemporaries, rests solidly and securely upon his amazing *Conic Sections*, a work in eight books, or, as we would say these days, in eight chapters, and of which the first seven have come down to us. Apollonius wrote a number of other works of lesser extent, but practically all of these are known to us only by their titles.

The Italian Renaissance aroused an all-consuming interest in the writings of the ancients. This eagerness embraced also the works of the Greek mathematicians, and in particular those of Apollonius. But copies of the works of the Great Geometer, either in the original Greek or in Latin, were practically nonexistent in the West. To remedy the deplorable situation and to fill the gap, a diligent search for copies in Arabic translation was set afoot. Those efforts were crowned with a considerable degree of success.

On the other hand, the mathematicians of the Renaissance period, under the pressure of this insistent demand, engaged in the indoor sport of restoring or reconstructing the missing works of the Greek geometer, basing their work on such extracts, quotations, allusions, and references to the works of the master as could be found in the available books of the writers of the classical and post-classical period.

In regard to such a restored work, two questions naturally come to mind: firstly, how meritorious is it as a book, in its own right, and secondly, how does it compare to its vanished prototype? Of the two questions, the latter is far the more intriguing, in spite of, or perhaps because of, the fact that it may be deemed unanswerable. History, however, has its own bag of tricks and uses them at times very shrewdly.

Early in the seventeenth century, the Dutch mathematician Willebrord Snell (1580–1626), better known as Snellius, worked on three of the smaller writings of Apollonius and succeeded in solving all the problems which those works were reputed to contain. It happened that some time later the original of one of those three works, namely *De sectione rationis*, came to light. Jean Étienne Montucla (1725–1799) expressed the following opinion (*Histoire de mathématiques*, Vol. 1, p. 263, Paris, 1758): "Il s'en faut beaucoup que l'ouvrage du Géomètre Moderne soit comparable à celui de l'Ancien." ("The work of the modern geometer is far from being comparable to the work of the ancient author.")

It can be argued that such an outcome should have been foreseen. Indeed, that a "run-of-the-mill" author would rival the creative genius of an Apollonius is too much to expect. Nevertheless, it would be

[1] Based on a lecture delivered in August, 1960, at the University of New Mexico, in Albuquerque, under the auspices of the National Science Foundation.

unfair to jump to the conclusion that restoration products of the Renaissance period were all of mediocre quality. This may readily be shown by a "counterexample."

At the time when only the first four books of Apollonius' *Conic Sections* were available, the restoration of the fifth book was undertaken by Vincenzo Viviani (1622–1703), a pupil of Galileo and a mathematician held in highest esteem by his contemporaries, but whose fame time has mercilessly tarnished. Viviani's effort was published in 1639, under the title, *De maximis e minimis geometrica divinatio in quintum Conicorum Apollonii Pergaei adhuc desideratum.* A year later appeared a Latin translation of the original Book V of Apollonius, made from an Arabic copy discovered shortly before. Montucla himself admits (*ibid.*, p. 262) that the comparison between the original and Viviani's restoration not only "is not disadvantageous to the Italian geometer," but that the latter's book is in some respects better than the work of the ancient master, and in many places supplements it quite effectively.

It may perhaps be appropriate to mention here still another source of ancient manuscripts, a source which, while less abundant, was often much more spectacular.

Before the invention of printing, and more particularly in antiquity, it was a fairly frequent practice to erase the writing from a parchment or other piece of writing material and to use the salvaged piece for a new work, although the original text was still faintly visible. Such a manuscript is called a *palimpsest* (*palin*, again; *psao*, rub smooth). It happened not infrequently that the hardly discernible old text was of much greater interest to later generations of readers than its more conspicuous successor and rival.

In 1906, the Danish classical philologist J. L. Heiberg (1858–1928) was invited to come to Constantinople (at present Istanbul) to examine an ancient palimpsest brought there from a monastery in Jerusalem. The faint text was written in Greek letters and contained diagrams; it was obviously a mathematical work.

Heiberg succeeded in restoring and deciphering nearly the whole of the old text. It consisted of parts of various works of Archimedes, and proved, among other things, that Archimedes was familiar with the center of gravity of figures in space—a much disputed historical question up to that time. But the thing that made this palimpsest a really great find is that it also contained the work of Archimedes known as *The Method*, a work which had been deemed irretrievably lost. The historical importance of this "method" is readily appreciated, for by it Archimedes anticipated by two thousand years the basic ideas of the calculus elaborated in the seventeenth century by Newton, Leibnitz, Fermat, and their lesser precursors and contemporaries.

Solutions of the problem of Apollonius by Viète and Romanus

Among those writings of Apollonius which we know only by the references made to them by other authors is the book *De tactionibus* (On tangencies), which, among other things, contained the following problem (the only difficult one in the book): To draw a circle tangent to three given circles.

François Viète (1540–1603) restored the book and published it under the title *Apollonius Gallus* in 1600. The restoration included a solution of the "Problem of Apollonius." (See Viète's collected works, pp. 325–338, issued under the title *Opera mathematica*, by F. van Schooten, Leiden, 1646.)

Viète's method consists in considering a number of special cases in which some or all of the given circles are replaced by points or straight lines. He uses those cases as steps toward the desired goal of solving the original problem. To solve the special cases he establishes a number of

auxiliary propositions or lemmas, in which he anticipates to some extent the theory of circles created two centuries later by Gaspard Monge and his pupils. For instance, Viète found and made use of the centers of similitude of two circles. The question of the number of solutions that the problem may have is not considered by Viète.

Viète's solution of the Apollonian problem was reproduced by Newton in his *Arithmetica universalis*, London, 1707, 1722 (English translation, London, 1707, 1720, pp. 143–148, Problems 37–41).

As a challenge, Viète proposed the problem to the Belgian mathematician Adrian van Roomen, known as Adrianus Romanus (1561–1615). The latter solved the problem by observing that if a circle (Z) is tangent to two circles (A) and (B), the difference between the lengths of the two lines of centers, ZA and ZB, is equal to the difference of the radii of (A) and (B). Hence the center Z of a circle (Z) tangent to three given circles (A), (B), (C) may be determined as a point common to two hyperbolas. To the great chagrin of Romanus, Viète declared the solution unsatisfactory on the ground that the use of conics in the determination of the center is inadmissible in the case of a problem of the kind considered; only a ruler and compass construction would do. Viète's opinion is shared by Montucla.

Newton's solution

In his great and famous *Principia*, Newton returns to the problem of Apollonius (Book I, Section IV, Lemma 16). He bases his solution on Romanus' idea and makes use of the properties of the hyperbola. But (Montucla, *ibid.*, pp. 262–264), "Ici Newton réduit avec une adresse remarquable les deux lieux solides de Romanus à l'intersection de deux lignes droites." ("Here Newton reduces with remarkable skill the two solid loci of Romanus to the intersection of two straight lines.") The "remarkable skill" of Newton is quite conspicuous in his solution, but the statement about the two straight lines is inaccurate, as will be pointed out presently.

Newton formulates the problem as follows (*Sir Isaac Newton's Mathematical Principles of Natural Philosophy and His System of the World*, translated into English by Andrew Motte in 1729, translation revised by Florian Cajori [1859–1930], pp. 72–73, University of California Press, Berkeley, California, 1947):

From three given points to draw to a fourth point that is not given three right lines whose differences shall be given or are zero.

Case 1. Let the given points be A, B, C, and Z the fourth point which we are to find; because of the given difference of the lines AZ, BZ, the locus of the point Z will be an hyperbola whose foci are A and B and whose principal axis is the given difference.[2] Let the axis be MN. Taking PM to MA as MN to AB, erect PR perpendicular to AB,[3] and let fall ZR perpendicular to PR; then from the nature of the hyperbola,[4] $ZR:AZ = MN:AB$. And by the like argument, the locus of the point Z will be another hyperbola, whose foci are A, C, and whose principal axis is the difference between AZ and CZ;[5] and QS a perpendicular on AC may be drawn,[6] to which (QS) if from any point Z of this hyperbola a perpendicular ZS is let fall, this (ZS) shall be to AZ as the difference between AZ and CZ is to AC. Wherefore the ratios of ZR and ZS to AZ are given and consequently the ratio of ZR to ZS one to the other;[7] and therefore if the right lines RP, SQ, meet in T, and TZ and TA are drawn, the figure $TRZS$ will be given in kind, and the right line TZ, in which the point Z is somewhere placed, will be given in position.[8] There will be given also the right line TA, and the angle ATZ; and because the ratios of AZ and TZ to ZS are given, their

[2] This is the difference of the radii of the two given circles, with centers A and B, of the Romanus solution.

[3] PR is the directrix of the hyperbola relative to the focus A.

[4] That is, from its definition in terms of its focus and the corresponding directrix.

[5] That is, the difference of the radii of the circles having centers A and C. Denote this difference by UV.

[6] QS is the directrix of the second hyperbola relative to the focus A, and we have $ZS:AZ = UV:AC$.

[7] We have $ZR/AZ : ZS/AZ = MN/AB : UV/AC$, or $ZR:ZS = (MN)(AC):(UV)(AB)$. Construct a right triangle with the mean proportionals of MN and AC, UV and AB as legs. The foot of the altitude from the right angle divides the hypotenuse in the desired ratio $ZR:ZS$.

[8] The lines TPR and TQS are given in position. If a point X is constructed whose distances from TP and TQ are in the given ratio $ZR:ZS$, the line TX passes through Z.

ratio to each other is given also;[9] and thence will be given likewise the triangle ATZ, whose vertex is the point Z.[10] Q.E.I.

Case 2. If two of the three lines, for example AZ and BZ, are equal, draw the right line TZ so as to bisect the right line AB; then find the triangle ATZ as above. Q.E.I.

Case 3. If all the three are equal, the point Z will be placed in the center of a circle that passes through the points A, B, C. Q.E.I.

The required point Z is finally obtained as the intersection of a line with a circle, and not as the intersection of two straight lines as Montucla states. But Newton has devised a ruler and compass construction which would have satisfied Viète.

Newton's solution of the problem of Apollonius was rediscovered, quite independently, by the Irish mathematician John Casey (1820–1891). (See *Sequel to Euclid*, p. 122, 1881.)

Newton himself seems to have attached considerable importance to ruler and compass solutions. Thus Lemma 19, Book I, Section V, of the *Principia* concludes with the following boast, "And so we have given in the Corollary a solution of the famous Problem of the ancients, begun by Euclid, and carried out by Apollonius, and this is not an analytical calculus but a geometrical composition, such as the ancients required."

Newton, like Viète, neglects to discuss the number of solutions that the problem of Apollonius may have. Perhaps the physical limitations imposed upon the solution by the applications Newton had in mind made such a discussion inconsequential. Newton uses his Lemma 16 in the Proposition 21 immediately following that lemma. In the scholium to that proposition he remarks, "When the conic is an hyperbola I do not include the conjugate hyperbola under the name of this conic. For a body going on with a continuous motion can never pass out of one hyperbola into the conjugate hyperbola."

As is readily concluded from the above remark, Newton used the problem of Apollonius for peaceful purposes, namely to determine the orbits of celestial bodies. The article devoted to Viète in the *Enciclopedia Italiana*, Volume 35, states that the same problem played an important role in the world war of 1914–1918; it was used to locate an enemy gun by listening to its sound at three different points. The American army seems still to be interested in this method.

Early cartesian solutions

The newly invented analytic geometry (1637) tempted, no doubt, many a devotee of the discipline to try this new powerful tool on the enticing old problem of Apollonius. Descartes himself found two such solutions. The first was so complicated that, according to the inventor himself, he would not undertake to carry out the resulting construction in a month's time. The second solution, while much simpler, was complicated enough to keep Descartes from touching it.

Among those who tried to solve this problem analytically was Princess Elizabeth (1596–1662), daughter of James I of England, and the wife of Palatine Frederick V, King of Bohemia. The princess and queen communicated her solution to Descartes, with whom she maintained a correspondence. But her solution was not any better than Descartes' own. Montucla concludes that the problem of Apollonius is one which does not lend itself to analytic treatment.

The unsuccessful attempts of Descartes and others to devise a workable solution of the problem of Apollonius using Cartesian co-ordinates has not deterred others from trying to solve the problem by algebraic calculations. Two such solutions were published in the *Memoires of the St. Petersburg Academy of Science* in 1788, one by no lesser a light than Leonard Euler (1707–1783), and the other by Nicolas

[9] If Y is the foot of the perpendicular XY from the point X (see footnote 8) of the known line TX upon the known line TQS, we have $TZ:ZS = TX:XY$. Now (footnote 6) $AZ:ZS = AC:UV$, whence $TZ:AZ = (XY:TY):(AC:UV)$.

[10] In triangle ATZ we know the base TA and the ratio $TZ:AZ$, which gives a circle as the locus of Z. A second locus is the known line TXZ.

Fuss (1755–1826), a pupil of Euler, and later perpetual Secretary of the Academy. However, a new dawn was already rising on the horizon of the problem of Apollonius.

Return to synthetic methods

The last quarter of the eighteenth century witnessed the revival of synthetic geometry, initiated by Gaspard Monge (1746–1818), the founder of descriptive geometry, and eagerly developed by his brilliant pupils and ardent followers. The circle was a favorite object of those studies. The theory of the circle was enriched by such new or renewed ideas as the power of a point for a circle, the radical axis of two circles, the orthogonal circle of three circles, the centers and axes of similitude of two and three circles, pencils of circles, etc. Each of these concepts individually, and all of them collectively, presented a temptation and a challenge to apply them to the problem of Apollonius.

These developments were all pretty much in the manner and the spirit of the classical geometry of the Greeks. Synthetic projective geometry, fathered by J. V. Poncelet (1788–1867), offered new methods of solving geometrical problems based on new theories, of which the theory of poles and polars was most directly applicable to the circle.

Moreover, the problem of Apollonius was the beneficiary of another kind of windfall. The achievements of the revived synthetic methods in geometry prompted the adherents of analytical procedures to refine and sharpen their tools, in which endeavor they were quite successful. The rivalry between the two schools during the first half of the nineteenth century was very keen and not always amicable. It happened that the problem of Apollonius played the role of a "test case" in the competition. This circumstance enhanced the interest in the problem.

The credit for having been the first, in this new epoch, to return to the problem of Apollonius goes to the poet, statesman, and mathematician Lazare Nicolas Marguerite Carnot (1753–1823). In his *De la corrélation dans les figures de géométrie* (Paris, 1801), he determines the radius of the required circle by quadratic equations (No. 158) and identifies the point of contact as a center of similitude (No. 159).

Carnot returns to this problem in his *Géométrie de position* (1803). On p. 390, §334, he outlines an algebraic method of solution. He does not carry out the calculations and does not offer a method of constructing the required circle. Viète, Newton, and Euler are named as forerunners. The name of Apollonius is not mentioned.

The great C. F. Gauss (1777–1855), the discoverer of a Euclidean construction of a regular 17-sided polygon, contributed a note to the German translation (1810) of Carnot's *Géométrie*, in which he completes Carnot's solution of the problem of Apollonius and simplifies Carnot's calculations as successfully as Newton had done with the solution of Romanus (Gauss, *Werke*, Vol. 4, p. 399).

In 1811, while a first-year student at the famous École Polytechnique, J. V. Poncelet contributed a solution of the problem of Apollonius [2; Vol. 2, p. 271].[11]

Four years earlier another solution of the problem had been published in the same journal [2; Vol. 1, 1806, pp. 193–195] by another "freshman" of the same school, Augustin L. Cauchy (1789–1857). This was the first article of the future prolific writer and leader of the school of rigor in mathematical analysis during the first half of the nineteenth century.

In spite of the fact that these solutions chronologically belong to the new era, and in spite of their intrinsic interest, they are nevertheless "old style." The new tools which the new era made available for the solution of the Apollonian problem increased the requirements which such a solution had to satisfy. Viète's method of

[11] Numbers in brackets refer to References at the end of the article.

solving the problem "on the installment plan," brilliant as it was for his time, was no longer acceptable. The new times required a direct method of treating the problem, and, furthermore, that the method should be applicable to all the special cases of the problem considered by Viète and others. A still further requirement was that the method of solution should in each case readily permit one to determine the number of solutions the problem may have, a question which had been ignored by practically all the earlier solvers, from Viète down.

Direct solutions

The first direct solution of the Apollonian problem was published by J. D. Gergonne (1771–1859), in the *Annales de Mathématiques* [3; Vol. 7, 1816–1817, pp. 289–303], of which journal he was the founder and editor. Gergonne arrived at his result by the analytic method.

A few years later, J. V. Poncelet (1788–1857) published another direct solution of the Apollonian problem based on considerations of pure geometry [6; p. 139].

Both of these solutions were greatly admired, but which of the two methods, the analytic or the synthetic, produced the better result was a much disputed question.

Nearly three quarters of a century later, Maurice Fouché undertook the study of the Apollonian problem by the synthetic method, making use of isogonal circles. In addition to a direct solution of the problem, he published a detailed discussion of the number of solutions the problem admits in all the cases presented by the various relative positions the three given circles may occupy with respect to each other. He also showed how his general direct solution may be applied to the special cases where some of the given circles are replaced by points or straight lines [5; 1892, p. 287]. However, Fouché's direct solution turns out to be basically the same as the one given by Poncelet.

The Fouché-Poncelet solution may be stated rather briefly, using somewhat more recent terminology.

Given three circles (C), (C'), (C''), let S', S'' denote centers of similitude of the pairs of circles (C) and (C''), (C) and (C'), respectively [1; §393 or §397], and let (S'), (S'') denote the circles centered at S', S'' and coaxal with the corresponding pairs of given circles [1; §446].

Let (V) denote the coaxal pencil of circles determined by the two circles (S'), (S'') [1; §441]. In the coaxal pencil (W), conjugate to the pencil (V) [1; §456], determine the two circles (D), (D') tangent to the circle (C) [1; §470]. The circles (D), (D') are two solutions of the Apollonian problem.

Indeed, the circle (D) belongs to the pencil (W); it is therefore orthogonal to (S') [1; §456]. On the other hand, (S') is a circle of antisimilitude of the two circles (C), (C'') [1; §482]; hence the line $S'T$ joining S' to the point of contact, say T, of the two circles (D), (C), meets (C'') in the antihomologous point, T'', of T on (C''), and the circle (D) touches (C'') in the point T'' [1; §526 b and d, §532 b, §544].

Considering the center of similitude S'', it may be shown in a like manner that (D) touches the circle (C') in the antihomologous point T' of T on (C').

In an analogous way, it may be shown that the circle (D') is tangent to (C') and (C'').

The line $S'S''$ is an axis of similitude of the three given circles, and considering this axis we obtained two solutions of the problem. Since three circles have, in general, four such axes [1; §406], the problem of Apollonius has, in general, eight solutions.

The last conclusion may perhaps seem somewhat hasty. In the construction of the circles (D), (D'), the role of the circle (C) is different from the roles played by (C') and (C''). If the role of (C) be assigned to (C'), would not such a change yield additional solutions, relative to the same axis of similitude $S'S''$?

The line $S'S''$ meets the line of centers $C'C''$ of the two circles (C'), (C'') in a center of similitude, S, of the latter circles [1; §§406, 407]. The circle of antisimilitude (S) of (C'), (C'') is coaxal with the circles (S'), (S''), because each of those three circles with collinear centers is orthogonal to the orthogonal circle (R) of circles (C), (C'), (C'') [1; §458]. Consequently, if the special role played by (C) in the construction of the circles (D), (D') were assigned to either of the other two given circles, say (C') instead of (C), the pencil (V) would remain the same, and therefore so would the pencil (W).

Using (C') we would have to determine the two circles (E), (E') belonging to the pencil (W) and tangent to (C'). Now the circles (D), (D') that we obtained before satisfy these conditions, hence (E), (E') are identical with (D), (D').

Thus there are two and only two circles corresponding to an axis of similitude of the given circles.

Conversely, any circle (D) tangent to three given circles (C), (C'), (C'') is related to an axis of similitude of those circles [1; §409]; hence the Apollonian problem cannot have more than eight solutions.

Number of solutions

The actual number of solutions of the Apollonian problem depends upon the relative positions of the given circles, as has already been mentioned. We shall briefly consider the following cases.

Case 1 (the three given circles taken in pairs have no points in common). There are eight solutions if the circles are external in pairs, or if two circles are external to each other and both are internal to the third. But if two circles are external to one another and the third circle is internal to one of them, the problem has no solutions.

Case 2 (two of the three circles intersect). (a) If the third circle cuts both circles but does not pass through either of their common points, there are eight solutions. (b) The problem will have four solutions if the third circle cuts only one of the first two, or does not cut either of them, or passes through one of their points of intersection. In the latter case, that point may be said to be a fourfold solution of radius zero. (c) If the third circle passes through the two common points of the first two circles there are no solutions other than those two points, each of which may be considered a fourfold solution of radius zero.

Case 3 (two of the circles are tangent to each other). The problem has six solutions if the third circle does not cut the first two, or when the contact of the first two circles is external and the third circle is external to both of them, or when the contact of the first two is internal and the third circle lies inside the larger one of them but outside the smaller one. In all other cases there will be four solutions. A solution passing through the point of contact of the first two circles would count for a double solution. There will be other double solutions if the third circle is tangent to one of the first two, or to both.

Case 4 (the three circles touch each other at the same point). The problem now has an infinite number of solutions [1; §445 c].

The interested reader may find a detailed discussion of these various cases in Fouché's article mentioned above, or in Rouché and Comberousse, *Traité de géométrie*, Seventh Edition, Volume 1, pp. 297 ff., Paris, 1910. Each of these publications also includes a complete exposition of the way the general solution may be applied to the cases where some of the given circles are replaced by points or straight lines.

The question of the number of solutions of the Apollonian problem was thoroughly discussed by R. F. Muirhead in his article "The number and nature of the solutions of the Apollonian contact problem," *Proceedings of the Edinburgh Mathematical Society*, Volume 14, 1895–1896, pp. 135–147.

More recent developments and extensions

The theory of inversion, developed during the first half of the nineteenth century, was utilized by Julius Petersen (1839–1910) to produce a very elegant solution of the Apollonian problem. A description is found in Problem 403 of Petersen's *Methods and Theories for the Solution of Problems of Geometrical Construction* (original Danish edition, Copenhagen, 1866; English translation, Copenhagen, 1879; reprint, New York, 1927).

But the progress of geometry not only provided new and admirable solutions of the problem; it transformed and generalized the problem itself.

If a circle is tangent to three given circles, then the angles that it makes with the given circles are equal, namely to 0° (or to 180°). The circle may therefore be said to cut the given circles "isogonally" (at equal angles). The Apollonian problem may thus be generalized by requiring the circle to cut the given circles isogonally, the angle of intersection being any given angle (Poncelet, Steiner).

The next, almost obvious, step is to require that the circle cut each of the given circles at a given angle.

The three circles, instead of being coplanar, could be taken on the same sphere. Carnot seems to be the first to have considered this extension; he outlines a computational way of finding the radius of the required circle [4; p. 415, §356]. J. D. Gergonne followed with another solution of this extended problem [3; Vol. 4, 1813–1814, p. 349]. Vanson solved the problem by elementary spherical geometry [5; 1855, p. 62]. The problem was also considered by many others, and was further generalized by replacing the sphere by a quadric surface and the circles by plane sections of that surface (Charles Dupin and others).

Extension to three-space

Within less than a century after Viète published his solution of the Apollonian problem, the following problem was formulated by Pierre Fermat (1601–1665), one of the founders of the modern theory of numbers: Construct a sphere tangent to four given spheres. The pattern of Viète's solution of the problem in the plane is the one Fermat followed in solving the problem in space (Fermat, *Varia opera mathematica*, p. 74, Tolos, 1679). Fermat's generalization of the problem of Apollonius is referred to by some writers as Fermat's problem. His solution was translated into French by Hachette in the *Journal de l'École Polytechnique* (cahiers 7 and 8, 1806). However, the first to call attention to the problem was Carnot [4; p. 357, §416]. He gives a set of equations which will yield an equation, presumably of the second degree, determining the radius of the required circle.

In 1808, Hachette published a synthetic solution of Fermat's problem [2; Vol. 1, p. 28]. Jacques Français published an analytic solution in 1809 [2; Vol. 2, pp. 63–66]. Charles Dupin contributed a synthetic solution in 1813 [2; Vol. 2, p. 423]. These writers were among the forerunners of a large number of authors who throughout the nineteenth century discussed Fermat's problem in books and mathematics periodicals, using both the analytic and synthetic methods, as for instance, Th. Reye, *Synthetische Geometrie de Kugeln*, Leipzig, 1879; J. A. Serret, "De la sphère tangente à quatre sphères données," *Journal für die reine und angewandte Mathematik*, pp. 51–57, Volume 37, 1848 (the journal founded by A. D. Crelle in 1826).

For a successful handling of Fermat's problem it was imperative to find for the sphere the geometric elements which played such an important role in the history of the problem of Apollonius. Thus the spheres were endowed with centers of similitude, radical planes, etc. They were grouped into pencils and nets, and considered in the cases where they are orthogonal to each other, or where some of them were cut by others isogonally, etc.

The parallelism between the plane and

space is in this case so striking that authors often were tempted to treat the problems of Apollonius and Fermat together. Thus George W. Oakley gives an analytic solution for both the Apollonian and Fermatian problems in *The Mathematical Monthly*, Volume 2, 1859–1860, pp. 116–126, and Benjamin Alvord discusses the two problems synthetically in his article "The intersection of circles and intersection of spheres," *American Journal of Mathematics* (founded by J. J. Sylvester), Volume 5, 1882, pp. 25–44 and four additional pages of figures.

Whether any given part of the theory of spheres was invented in order to solve the problem of Fermat, or that theory derived its inspiration in another way and then happened to be of use in connection with this problem, is very difficult to tell. But if any problem may be said to have dominated the development of a geometric theory, the problem of Apollonius and its generalization by Fermat is such a problem in connection with the geometry of the circle and the sphere, and perhaps of a good deal more beyond that.

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A history-of-mathematics time strip

by Howard Eves, University of Maine, Orono, Maine

I have found the following device useful in the classroom. I took a long roll of paper (3.5 inches wide and such as is used in some adding machines) and inserted one end in the typewriter. Then, letting each turn of the typewriter roller represent five years, I typed the following list of dates and events, putting each item, however, at its proper time position on the roll. The dates were typed at the extreme left of the strip, and each date was followed by its associated brief description (if it has one) but separated from the dates by ten typewriter spaces. The space on the strip between the dates and the descriptions was used for a purpose to be described later. I left about 8 inches of clear paper before starting to type the list of dates and events. It is understood that a minus sign before a date indicates a B.C. date.

−3000	Egyptian reliefs referring to taxes
−2900	Great Pyramid built
−2800	
−2700	
−2600	
−2500	
−2400	Babylonian tablets of Ur
−2300	
−2200	Mathematical tablets at Nippur
−2100	
−2000	
−1900	
−1850	Oldest extant astronomical instrument; Moscow papyrus
−1800	
−1700	
−1650	Rhind papyrus
−1600	
−1500	Oldest Egyptian sundial
−1400	
−1350	Later Nippur tablets; Rollin papyrus (bread problems)
−1300	
−1200	
−1180	Harris papyrus (temple wealth)
−1100	
−1000	

−900
−800
−700
−600 Thales (demonstrative geometry)
−540 Pythagoras
−500
−460 Hippocrates of Chios (quadrature)
−450 Zeno (paradoxes)
−430 Antiphon (exhaustion)
−425 Hippias (quadratrix)
−400 Archytas (proportion)
−380 Plato (academy)
−370 Eudoxus (proportion)
−350 Menaechmus, Dinostratus
−340 Aristotle (logic)
−300 Euclid
−230 Eratosthenes
−225 Apollonius, Archimedes
−200
−180 Nicomedes, Diocles (trisection, duplication)
−140 Hipparchus (astronomy)
−100
0
75? Heron, Diophantus
100 Nicomachus, Menelaus
150 Ptolemy (astronomy)
200
300 Pappus
390 Theon (commentator)
400
410 Hypatia (first woman mathematician)
470 Proclus (commentator)
500
510 Aryabhatta the Elder
600
630 Brahmagupta
700
710 Bede (finger reckoning)
800
820 Al-Kowarizmi
850 Mahavira
870 Tabit ibn Qorra
900 Abu Kamil
980 Abû'l-Wafā
1000
1100 Omar Khayyám
1150 Bháscara
1200 Fibonacci
1225 Nemorarius
1250 Roger Bacon, Nasir ed-din
1300
1360 Oresme
1400
1450 Nicholas Cusanus (or N. of Cusa)
1460 Peurbach
1470 Regiomontanus
1494 Pacioli
1500 Leonardo da Vinci
1510 Albrecht Dürer
1520 Copernicus
1525 Stifel, Rudolff
1542 Robert Recorde
1545 Ferrari, Tartaglia, Cardano
1550 Rhaeticus
1572 Bombelli
1580 Vieta
1590 Cataldi, Stevin
1595 Pitiscus
1600 Harriot, Galileo
1610 Kepler
1615 Napier, Briggs
1630 Mersenne, Oughtred
1635 Fermat, Cavalieri
1637 Descartes, Desargues, Roberval
1650 Pascal, Wallis
1670 Barrow, Huygens
1680 Newton, Leibnitz
1690 l'Hospital, Jacques Bernoulli
1700 Jean Bernoulli
1710 Roger Cotes
1720 Taylor, DeMoivre
1740 Maclaurin
1750 Euler
1760 d'Alembert
1770 Lambert
1780 Lagrange
1800 Gauss, Laplace, Legendre
1806 Argand
1815 Fourier
1825 Abel
1830 Lobachevski, Bolyai, Poncelet, Cauchy
1840 Steiner
1845 Hamilton, Grassmann
1850 Staudt, Chasles, Liouville
1855 Cayley, Plücker, Riemann, Dase
1900
1950
2000

The completed strip, with its 8-inch ends, measures almost 15 feet in length, and when taped to the classroom blackboard it runs pretty well across the room.

In the 8-inch leader preceding the list of dates, I typed a title, the information that one turn of the typewriter roller represents five years, and a few important very early dates such as: −4700 (possible beginning of the Babylonian calendar), −4241 (Egyptian calendar introduced), −4000 (metal discovered), −3500 (writing in use).

Now I will describe what I did in the space on the strip between the dates and their descriptions. Starting at −600, I drew a line in the left side of this space, parallel to the strip and terminating at about 515. This was labeled "The Greek Period." I drew a second line, a bit further to the right, running from about 505 to about 1160, and labeled this line "Hindu Mathematics." A third line was drawn from about 630 to about 1110 and was

labeled "Arabian Mathematics." A fourth line from about 515 to 1150 was drawn and labeled "The Dark Ages." A fifth line (occupying the position of the first line) was drawn from about 970 to 1600, and the last four inches of this line were made dotted; this line was labeled "Period of Transmission." Finally, a sixth line (occupying the position of the second line) was drawn from about 1470 to the end of the strip and was labeled "The Modern Period." The idea of drawing these lines was suggested by the chart just preceding the index in Vera Sanford's *A Short History of Mathematics.*

The time strip shows at a glance the relative lengths of the important periods in the history of mathematics. One sees how long (almost half the strip) was the pre-Hellenistic period, and how sparse and scattered is our information concerning this period. The long blank spaces coincide with particularly turbulent periods of history. One sees the condensation of events in the early part of the Greek period, followed by a period of gradual dissolution. The extreme condensation and shortness of the modern period is striking to observe. One sees how, during Europe's Dark Ages, mathematics made a detour through India and Arabia, finally returning to Western Europe during the period of transmission.

It is clear that with a wider strip more descriptive information could accompany each item. It is also to be noted that similar strips can be made for, say, just the history of geometry, or of algebra, or of trigonometry, or of special topics like the computation of pi, etc.

Letters to the editor

[*Editor's note:* Readers who are interested in problems, such as those in the article by Lewis and in this letter by Scott, may find very adequate treatments in many textbooks, especially those on number theory. This letter is published in the interest of acquainting more teachers with the possibilities of this kind of problem for use in their classes.]

Dear Editor:

In the November, 1960, issue of THE MATHEMATICS TEACHER, Mr. Lewis presented "An Interesting Problem Involving Indeterminate Equations." This brought to mind another problem involving three unknowns and which provided only two equations. However, certain other conditions are known and, therefore, one can find a unique solution. It is not always necessary to have as many equations as one has unknowns. However, it *is* necessary to have sufficient given conditions. The following will illustrate.

The problem may be stated as follows: A farmer has \$100 and wishes to buy exactly 100 head of livestock, including at least one cow, one hog, and one sheep. He must pay \$10 for each cow, \$3 for each hog, and 50 cents for each sheep. The two equations arising from this statement are:

$$x+y+z=100, \quad (1)$$

$$10x+3y+\tfrac{1}{2}z=100, \quad (2)$$

where x represents the number of cattle, y the hogs, and z the sheep. Under ordinary conditions these equations would produce no unique solution. However, there are certain other conditions implied in the problem which, along with equations (1) and (2), will permit a unique solution. These will be stated as their need arises.

Multiply (2) by 2 and subtract equation (1) from it.

We have:

$$19x+5y=100 \quad (3)$$

Solve (3) for y:

$$y=20-\frac{19x}{5} \quad (4)$$

From the statement of the problem we know that x, y, and z are natural numbers. From (4), then,

$$x\equiv 0 \mod 5 \quad (5)$$

The problem implies a maximum limit on x (number of cattle). If the farmer purchased 10 head of cattle his \$100 would have been expended and no hogs or sheep would have been obtained. Since he must buy at least one of each kind of livestock, then $[1\leq x\leq 10]$, therefore, from (5), $x=5$. And from (4), if $x=5$, $y=1$. Then, since $x=5$ and $y=1$, from (1), $z=94$. We see that this triple satisfies equation (2).

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Reviews and evaluations

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BOOKS

Algebra Problems, Donald S. Russell (New York: Barnes and Noble, Inc., 1960). Paper, vii+134 pp., $1.75.

This book is one of a series of paperbacks in many subject areas published by Barnes and Noble.

In contrast to most algebra books which present a developmental approach for students studying algebra for the first time, this book has a different purpose. Its subtitle is *A Portable Blackboard of Basic Principles and Complete Step-by-Step Solutions.* The author has included the material which would appear on a blackboard in a competently taught course of elementary and intermediate algebra. Rather than having the explanation erased from view as happens in the classroom, students with this book have a permanent record of these solutions of key problems. All exercises have complete solutions given. There is a large variety of problems to illustrate each new concept, more than eight hundred problems in all.

The content closely parallels that of traditional textbooks of elementary and intermediate algebra. A minimum of detailed explanation is given, but most of the key steps are explained in each problem. If a student is capable of comprehending a written sequence of mathematical material, he should be able to understand the reasons for the steps in the solution of the problems given. However, many students will have difficulty following the step-by-step procedure when reasons are not given.

No attempt is made to introduce any modern algebra in the book. This may be a serious drawback in many schools.

In the section on graphing, no mention is made of the slope concept. The ideas related to slope are usually presented in intermediate algebra because of their contribution to understanding the nature of equations as well as for their importance in later mathematics.

Verbal problems are included as a part of the chapters concerned with equations. In addition to this treatment, the last chapter deals with classification of word problems, such as age, mixture, and uniform motion. This chapter may be particularly valuable to students who need to see the common thread that runs through different types of problems.

Several groups of students might benefit from this book. Students currently studying algebra may find it helpful as a supplementary reference and permanent record of problem solutions. Actually, much of the material can be found in the explanatory pages of the regular textbook. However, the presentation here has been condensed and systematized, and the lack of descriptive material may be a help to some students. Fuller explanations must be presented from other sources to assure a satisfactory degree of understanding.

Students who have completed their mathematics study may find this a particularly helpful source for review purposes. With the increasing importance that is being put on competitive examinations for college entrance, the main value of the book may well be its use as a review of fundamental algebra by students who have not studied algebra for a period of time. For students who have a good background, this should be a good, concise reference for review purposes.

The publishers also recommend *Algebra Problems* for anyone learning algebra on a self-study basis. This use is questionable, since there are no problems for the student to solve. The value of the book would be increased if some unsolved problems were included at the end of each chapter. Students often do not know whether they really understand a new idea until they attack problems on their own.

Properly used, this book may be a help for some students. Its value seems to be either as a supplementary resource while studying algebra or as a reference for review purposes after completing the course.—*Arthur Cassell, Mount Vernon High School, Mount Vernon, Ohio.*

Applied General Mathematics, second edition, Edwin B. Piper, Randolph S. Gardner, and Joseph Gruber (Cincinnati: South-Western Publishing Company, 1960). Cloth, x+566 pp., $3.40.

This text and its accompanying booklets—*Manual, Solutions, Problems and Drills,* and *Tests*—form a planned and co-ordinated set of materials. Many organizational aspects of presenting a general mathematics course, such as suggested time schedules, plans for instructional procedures, and specific teaching suggestions, are so precisely described in the *Manual* and *Solutions* booklets that the time needed by a teacher for these preparatory steps may be kept to a minimum. The most inexperienced teacher using these materials

will feel secure. However, to use this prearranged program effectively, a teacher should study the *Manual* each day prior to entering the classroom.

The textbook contains sixteen units of work. The first ten units deal with consumer mathematics of the type found in most general mathematics texts. The authors strongly urge the teacher to present these ten units in the exact order given; the material presented in them follows a precise, graded sequence. Often new topics represent a more complex form of previously studied topics. Some problems require that the student review topics that appeared earlier. The remaining six units provide flexibility, since, with the exception of the unit on scale drawing and indirect measurement, they may be used in any order desired.

The mathematical content of the units is presented in a manner that fosters the mechanical or answer-producing type of response from the student. This approach is reinforced by problems in tabular form which are presented periodically. There is little or no attempt to strengthen or build the basic mathematical understandings into a strong, meaningful structure. Each mathematical operation is presented in a situational setting whose atmosphere tends to cloud its relationship to other mathematical operations. For example, the material illustrating the relationship between division and common fractions is buried in a section entitled "Purchasing a Fraction of a Unit." The explanation of subtraction and division makes no mention of the fact that these are the inverse operations of addition and multiplication, respectively. The inverse relationship between multiplication and division is called to the attention of the teacher in the *Manual* with the comment that this idea will aid the students in understanding division. The students must form the inverse concept from the following definition: "Division may also be defined as finding one of two factors when their product and one of the factors are known."

The units dealing with mensuration, algebra, geometry, scale drawing and indirect measurement, and set theory are tacked on at the end of the text and are a complete departure from the situational approach used in the previous units. The business-consumer theme which gives continuity to the earlier sections of the text is completely missing. The answer-producing and mechanical response type of presentation continues, however. Of the last six units, only those on algebra and weights and measures are included in the suggested time schedule for a ninth-grade course. The entire text is suggested for an eleventh- or twelfth-grade course.

The strongest attribute of this text is that together with the available supplementary publications it forms a complete working course for a teacher. No doubt many school systems will find that the content of this text meets the requirements of their general mathematics syllabus.

It is regrettable that the primary purpose of this course—fostering an understanding of the basic mathematical ideas—is made subservient to the development of business procedures.—*J. J. Brown, Washington, D.C.*

High School Mathematics: Unit 5, Relations and Functions, University of Illinois Committee on School Mathematics (Urbana, Illinois: University of Illinois Press, 1960). Paper, student's edition vii+278 pp., $1.50; teacher's edition, $3.

Here is a book which is *unique* among high school texts in mathematics. Other textbook writers have begun to emulate many of its ideas, and well they might, for the UICSM (University of Illinois Committee on School Mathematics) staff has written a work which will have far-reaching consequences.

At least two factors make this book significant. First is the selection of topics for a semester's study in the second or third year of high school. Through a clever introduction to the unit, the student becomes reacquainted with, and learns new notation in, working with sets, and is led to see that a relation is a set of ordered pairs. Early in the unit the algebra of sets is studied to give students familiarity with concepts they must use throughout the unit.

Attention should be called to the fact that use of this text presupposes a previous study of Units 1 through 4 written by the UICSM staff. These units give the student an understanding of and familiarity with the notion of sets, the graphing of equations and inequations, and the derivations of numerous generalizations about real numbers as a basis on which to build new concepts.

The notions of domain, range, and converse, and the properties of reflexivity and symmetry are introduced in the study of relations. The major part of the book deals with functions, a function being a relation of a special kind, namely, a set of ordered pairs with no two of the ordered pairs having the same first component. (In recent years this definition has been common for the word "function.") The concept of function is important for studying the remainder of the unit, which includes such topics as variable quantities, functional dependence, linear and quadratic functions; it is also important for studying the following units of the UICSM program. The groundwork laid in this unit in the study of functions and inverses of functions makes the teaching of logarithms and topics in trigonometry far easier. Such traditional topics as variation, systems of equations, completing the square (lots of work with this), and the quadratic formula are taught in this unit.

The other outstanding feature of this book is the careful way in which concepts are developed. The "discovery" approach to teaching is built right into the text exposition and exercises with more care than in texts by any other authors. There is considerable exposition which students are expected to read and un-

derstand—and they generally do. There is one place, pages 90 and 91, where the discussion is rather heavy, but the authors warn teachers of it, so that this can be circumvented if desired. The teacher will find an abundance of exercises for the kind of approach which creates an awareness of the concept for the student without his having to be told the rule first. There are also numerous review exercises to maintain skills, and optional exercises to interest and challenge the better students.

On page 81 more exercises are needed than are presently provided. Here students are to describe the inverse of those functions which have inverses, and more practice is definitely needed.

The student's edition is paper bound, while the teacher's edition consists of both the student's edition in loose-leaf form and green loose-leaf commentary pages, numbering more than 300, for the user to integrate with the white pages. These commentary pages contain answers to the exercises, helpful suggestions and background for teaching the material, and quizzes which the teacher may use.

This unit may be taught either following the study of Unit 4, which would be in the second year; or following the geometry unit, Unit 6, which would be in the second or third year; or partly in each, depending on the pace at which the class is taught. The reason for the option to take either Unit 5 or Unit 6 first is that school systems may often wish to have the teaching of geometry completed by the end of the tenth grade.

This reviewer has been using Unit 5 in its present form during the 1960–61 school year, as well as the preliminary editions in past years, and he finds this unit extremely interesting and stimulating to teach. One might think that teaching functional dependence in the manner in which it is done and teaching composition of functions to sophomores, or juniors, is expecting too much of the students (or even of the teacher), but here again the careful development in the text makes the learning (and teaching) relatively easy and quite intelligible. One helpful device is the use of many finite sets, each containing just a few ordered pairs, so that each ordered pair can be investigated individually, thus firmly implanting the concept. Knowledge of these topics helps to clarify and put new meaning into formulas, and it allows the authors to explain other concepts meaningfully.

If the teacher has acquired the inductive method of teaching and will lead the student to discover generalizations for himself, and if the teacher is well versed in the content of the first four units, he will find Unit 5 very teachable—delightfully so.—*Howard Marston, The Principia School, St. Louis, Missouri.*

Intermediate Algebra, Roy Dubisch, Vernon E. Hawes, and Steven J. Bryant (New York: John Wiley and Sons, Inc., 1960). Cloth, xii+286 pp., $4.50.

The topics covered in this text are those of a traditional course in intermediate algebra. They are well presented in a concise manner and with evidence that the authors are aware of other more modern approaches being made in some presentations.

There is a tendency toward developing the work as a logical system through the acceptance of definitions and axioms followed by proofs of theorems. The development of new work is constantly being shown through applications of fundamental laws. The basic laws of arithmetic are extended to cover algebraic numbers. Multiplication, division, and factoring of polynomials are the result of using the commutative, associative, and distributive laws of multiplication and division. The work with algebraic fractions is developed as a system of rational numbers. Stressing the principle that multiplying by 1, or by a fraction equivalent to 1, does not change the value of a number would seem to eliminate some of the common errors made by students working with algebraic fractions.

In presenting methods of solving systems of linear equations, the authors have covered the usual methods of graphing, substitution, and addition or subtraction. The graphing includes an understanding of slope, and through that concept two linear equations are proved to have no common solution, one solution, or an unlimited number of solutions. The writers have not stopped here but have taken the opportunity to introduce work with determinants of the second order and also with third-order determinants through the use of minors. It seems that the double-subscript notation ($a_{2}x_{2}$, $a_{1}x_{1}$) might confuse students who up to this time have had little work with subscripts of any kind. Perhaps such notation is justifiable because of the convenience in working later with determinants of higher order or with matrices, which are briefly introduced.

Geometric representations help to clarify topics. The real number system is represented on a line. Pairs of numbers are described geometrically as points in a plane. Graphs of equations in two variables are explained as lines in the Cartesian plane, and common solutions are thus shown. No reference is made to geometric representation in space. Students who have considered space concepts in their study of geometry will notice such omissions.

The function concept receives good consideration and with it the subject of inequalities. Equations are studied as "families of curves" when considering their graphs. The more common expressions of variation, "y varies directly as x," "y varies inversely with x," and "y is inversely proportional to the square of x," also fit into the family of curves. The theory of logarithms is shown as the theory of exponents. Numbers are written in scientific form for the work in logarithms; i.e., as a number between 1 and 10 multiplied by the needed power of 10. As a basis for the work in science, this is good, but it might prove

difficult for students who are being introduced to logarithms.

The author frequently cautions the reader as to the value of checking work. Results depend upon restrictions placed on the kind of number being used, as well as upon accuracy in computation. Letters, for example, cannot be given values which make a denominator zero. Work with complex or imaginary numbers is not included until the final unit. The summation symbol, $\sum$, is introduced in the work with sequences.

There are no sections devoted to accumulative review. Little opportunity is given students to explore possibilities and reach conclusions by themselves. The format of the book is good. Definitions and theorems are usually emphasized with heavy type. Some of the formulas, however, have not been identified that way. The number of word problems seems limited, although frequent references are made to the work in related subjects. More applications of the processes studied would emphasize the need for better understanding. Numerous exercises require proofs, thus showing that proofs are a part of algebra as well as of geometry.

For the student who has tried to "make sense" of his former work, this somewhat brief presentation of intermediate algebra should bring a much clearer understanding of the fundamental concepts of the subject.—*Margaret A. Striegl, East High School, Wauwatosa, Wisconsin.*

Mastering Elementary Algebra, Wallace Manheimer (New York: Oxford Book Company, 1960). Paper, xix+392 pp. $1.25.

Mastering Intermediate Algebra, Lester W. Schlumpf (New York: Oxford Book Company, 1960). Paper, xviii+357 pp., $1.25.

The author, in his preface, says, "Current thinking stresses the logical coherence of algebra and directs the student toward an appreciation of a postulational system. *Mastering Elementary Algebra* embodies this attitude, but tempers it with a realistic awareness of the level of understanding of the ninth-year student." He also states that so-called motivational material has been almost entirely omitted in the hope that a clear and cogent presentation of the subject, with a wealth of suitable exercises, will in itself constitute the best motivation.

This text does contain a wealth of exercises —many more than the usual text contains. Each set, moreover, is preceded by well worked-out examples. These examples are in well-arranged steps with a full explanation. There are a very large number of verbal problems of all kinds. The examples preceding these are fully explained and accompanied with tables and drawings which are unusually good. According to the index of exercises, there are 42 pages devoted to verbal problems.

At the end of each chapter are well-constructed tests. These are varied, consisting of completion, true-false, matching, multiple choice, and straight problem-solving.

This reviewer could find little evidence that this text directs the student toward an appreciation of a postulational system. The author introduces each new idea with a very brief intuitive explanation and then hurries to a rule covering the new idea. It seems to the reviewer that this text outdoes most traditional texts in the number of rules and mechanical aids presented. For example:

Page 26. One way to remember the order of operations is by means of this sentence. Please Remember My Dear Aunt Susie (Powers—Roots—Multiplication—Division—Addition—Subtraction).

Page 60. Transposition defined and used in solving equations.

Pages 245–46. A short-cut method for combining two fractions.

Cross products: ad, bc

Product of denominators: bd

Answer: $\frac{ad+bc}{bd}$

Page 255. Cross-multiplication, a useful short cut.

This process, called *cross-multiplication*, is most useful in solving simple fractional equations.

Page 331. The fundamental law of radicals.

The fundamental law of radicals is: *The square of a radical is equal to the radicand.* (This text makes no mention of radicals other than those indicating square root.)

This sort of treatment will be welcomed by students who have been brought up on traditional texts and who are always looking for a rule. It will not be welcomed by a good many teachers who try to emphasize thinking rather than memorizing.

The commutative, associative, and distributive properties of our real number system are discussed but not used extensively through the text. The distributive property is not mentioned in connection with factoring. The real number system is not discussed or even mentioned in the text. The word "integer" is used first on page 74 in connection with consecutive integers. Rational numbers are first mentioned on page 329 in the chapter on radicals. The number line is introduced early and used often.

There is a good chapter on inequalities starting on page 376. In this last chapter in the book, sets are mentioned casually a few times but not formally introduced or discussed. With all of the research that has been going on in the field of high school mathematics, one would have hoped for a very early introduction of these ideas and a simultaneous development of equalities and inequalities. This chapter, of course, can be introduced early in the year by the teacher, but this is not nearly as satisfac-

tory as having the ideas of the chapter permeate the whole development.

The word "function" is used interchangeably with "formula" throughout the chapter on variables and functions (pages 1–38) and then is not mentioned again until pages 293 and 296 in connection with direct and inverse variation, and on page 310 and the following pages in connection with trigonometry.

For the teacher who wants a traditional elementary algebra text with a very adequate supply of exercises, adequate illustrations, a simple statement of each new idea with well worked-out, step-by-step examples, and a rule for everything, this reviewer recommends *Mastering Elementary Algebra*. There are no extra colors used, a feature which makes some high school texts more attractive, but the very modest cost ($1.25) is a decided asset. School boards have so much invested in books these days that it is a real problem to discard one expensive text in favor of another just as expensive.

The teacher who wants students to share in formulating rules as a result of a slow intuitive and inductive approach to new ideas, who wants students to be familiar with the structure of the real number system, who wants ninth-grade students to have an introduction to a deductive system, who wants elementary algebra to develop equalities and inequalities side by side, and who wants algebra to be visibly based on the properties of the real number system—this teacher should look farther.

Mastering Intermediate Algebra is written in much the same manner as *Mastering Elementary Algebra*. There are many, many exercises following examples explained carefully in step-by-step arrangement. Either of these texts could be successfully used by a good student brought up on traditional texts, with little or no help from the teacher. *Intermediate Algebra* differs from *Elementary Algebra* in not having tests at the end of each chapter. The usual topics found in most second-year algebras are treated.

The first 130 pages of the intermediate text are almost purely a review of work covered in *Elementary Algebra*. The terms "rational" and "irrational" are mentioned first on page 116. The terms "real number" and "imaginary number" are mentioned first on page 134. The number line is shown and mentioned as a "number scale" on page 1, but never again mentioned.

The last chapter in the book is titled "Sets and the Logical Structure of Algebra." Here is found an adequate discussion of sets, the properties of the real number system, relations and functions. None of these ideas is mentioned earlier in the text. Inequalities are also treated in the last chapter, with little or no mention before. A good many of those interested in the teaching of secondary mathematics feel that the materials of this chapter should be basic to the whole algebra course.

Mastering Intermediate Algebra is a suitable text for students brought up on traditional mathematics texts. It would not be suitable for those brought up on the kind of materials developed by such groups as the School Mathematics Study Group.—*J. V. Naugle, Wyoming High School, Wyoming, Ohio.*

Have you read?

"A Conference on Mathematics Curricula in Institutes," *The American Mathematical Monthly*, January, 1961, pp. 33–38.

Many of you have attended and most of you have heard of the National Science Foundation Institutes for High School Teachers. This is a report of a conference called by the foundation to study progress to date. It will be of interest to note that in 1953 there were two institutes and in 1960, 396 summer institutes, 33 year institutes, and 204 in-service institutes. The purposes were to improve mathematical instruction of students and the competence and backgrounds of teachers.

Some of the guide lines for institutes are that they should be sequential, deal with teachers of one level at a time, and include in a group those of similar mathematics background. The quality of staff is measured by knowledge of mathematics, ability to communicate to students, and awareness of mathematics taught in secondary schools.

The courses should start with algebra, with emphasis on number systems and principles. Experimental work should be studied. Geometry should come next, but it should extend to reasoning and other geometries. Analysis includes essential concepts of calculus. Probability and statistics should come after algebra. Foundations should be limited.

Participants should have opportunities to discuss secondary-school curriculums. Graduate credit applicable to degrees should be provided.

This articles gives an excellent overview of the place of the institute in our present educational structure. Be sure to see the report on page 85. You may be surprised!—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

• TIPS FOR BEGINNERS

Edited by Joseph N. Payne, University of Michigan, Ann Arbor, Michigan, and William C. Lowry, University of Virginia, Charlottesville, Virginia

Let's say what we mean

by Edith Robinson, Wisconsin High School, Madison, Wisconsin

All of us who teach mathematics hope that our students will learn to think more clearly, yet our own generalizations sometimes lack the precision which would best foster clarity of thought. Is there anyone, for example, who has not at one time or another said to his class, "Any number divided by itself is 1"? We don't really mean this, but if a student takes us literally, he may conclude that 0/0 must be 1.

Or we may say today, "The square root of 25 is ± 5," and tomorrow ask the class to simplify $\sqrt{36}-\sqrt{25}+\sqrt{4}$. Acting on what we have said, the student could get

$$\sqrt{36}-\sqrt{25}+\sqrt{4}=(\pm 6)-(\pm 5)+(\pm 2)$$

from which he may have

$$(+6)-(+5)+(+2)=+\ 3$$
$$\text{or}\quad (+6)-(+5)+(-2)=-\ 1$$
$$\text{or}\quad (+6)-(-5)+(+2)=+13$$
$$\text{or}\quad (+6)-(-5)+(-2)=+\ 9$$
$$\text{or}\quad (-6)-(+5)+(+2)=-\ 9$$
$$\text{or}\quad (-6)-(+5)+(-2)=-13$$
$$\text{or}\quad (-6)-(-5)+(+2)=+\ 1$$
$$\text{or}\quad (-6)-(-5)+(-2)=-\ 3$$

Now, clearly, when we ask a youngster to simplify $\sqrt{36}-\sqrt{25}+\sqrt{4}$ we expect him to get one and only one answer; namely, +3. We don't want his answer to be -13 or $+9$; in other words, we expect him to use only the positive square roots. Then should we not, for the sake of clarity, say, "25 has two square roots, +5 and -5," and refer to them separately as the positive square root of 25 ($\sqrt{25}$), and the negative square root of 25 ($-\sqrt{25}$)? After all, we don't want someone in geometry to say that the measure of the hypotenuse of a given triangle is ± 24.

Or consider the matter of cancellation. We don't want the student to do this:

$$\frac{x+6}{2}=\frac{x+\cancel{6}^{3}}{\cancel{2}}=x+3.$$

Yet we often allow him to do this:

$$\frac{a^2-b^2}{a-b}=\frac{(a-b)(a+b)}{(a-b)}=\frac{\cancel{(a-b)}(a+b)}{\cancel{(a-b)}}=a+b.$$

When the student asks us why he can't cancel the 2 into the 6 in the first instance, we may reply that $\frac{x+6}{2}$ does not equal $x+3$ (it does when x is replaced by 0), or we may say that this is not "proper" cancellation. Is the cancellation "proper" in the second case? Suppose $a=b$; then we are dividing both numerator and denominator by 0 which is certainly not proper! What is "proper" cancellation?

Many ninth-graders (and indeed thirteenth-graders) have lost track of the true meaning of cancellation; they just cross things out. We can help by stating more explicitly what we are doing: "We may divide both numerator and denominator of $\frac{(a-b)(a+b)}{(a-b)}$ by $(a-b)$ provided $a-b\neq 0$."

In the case of $\frac{x+6}{2}$, we can't really say what this is equal to since we don't know what number x "stands for." If we need to, however, we can get an equivalent expression which is true no matter what numeral we put in place of x if we first note that dividing by 2 is the same as multiplying by $\frac{1}{2}$, and then use the distributive law:

$$\frac{x+6}{2}=\frac{1}{2}(x+6)=\frac{1}{2}\cdot x+\frac{1}{2}\cdot 6=\frac{x}{2}+3.$$

Similarly, in dealing with quadratics such as $x^2=4$, one is tempted to oversimplify. While it is intuitively obvious that if $x^2=4$, x must be $+2$ or -2, we don't want to summarize by saying "Therefore $x=\sqrt{4}=\pm 2$." A solution by factoring prevents such an easy generalization. Since $a=b \leftrightarrow a-b=0$, if $x^2=4$, then $x^2-4=0$ and hence $(x-2)(x+2)=0$. Not only does such a treatment point up two important properties of the real numbers, but it discourages the student from hunting for easy methods of solving quadratics which might prompt him to "solve" $x^2=2x$, for example, by dividing both sides by x.

Zero is indispensable as a number, besides being the additive identity for the real numbers, it is the reference point in the construction of the "number line." Yet students have the strangest notions about it. Some think it is "nothing" and hence not a number at all. Some believe that $5/0=0$; others that $5/0=5$; while still others say $5/0=\infty$.

If we are careful to make statements such as, "Any real number, except 0, when divided by itself is 1," it may serve as a reminder that zero is a number with unique properties. We ought to spend more time discussing these properties with our students.

Why does $0\cdot a=0$ for any real number, a? Why is division by 0 not allowed? Why is 0/0 undefined? We can, and should, give a good intuitive argument in answer to these questions.

Precisely what do we mean when we say that $24\div 8=3$? Does not the problem, $24\div 8=__$, ask the question $8\times__=24$? Then the problem, $5\div 0=__$, asks the question, $0\times__=5$. But there is no real number which, when multiplied by 0, gives 5; hence 5 cannot be divided by 0. Similarly, if we try to divide any other real number (except 0) by 0, we encounter the same logical difficulty.

The notion that $5/0=\infty$ probably arises in the following way: consider $5/x$, and choose a large value for x (say $x=100$); then $5/x$ is very small. As x gets smaller and smaller, $5/x$ gets larger and larger; in particular, when x is less than, say, .001, $5/x$ is very large. So, the argument goes, if $x=0$, $5/x$ must be infinitely large. But in propounding this argument, we have considered only positive values for x. Zero is not the smallest possible replacement for x. Looking at negative replacements for x, we find that as $x\to 0$, $5/x$ gets "infinitely" *small*. The graph of $y=5/x$ is a hyperbola, and the logical difficulty of assigning a numerical value to $5/0$ is intuitively obvious from such a graph.

So, let's watch our language; let's be more explicit about what operations we are performing and more precise about stating what numbers we are talking about. Is it always true that $\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$? Is it true that x^2+4 *cannot* be factored? Do "two negatives" *always* "make a positive"? If we as teachers set a good example, then we may hope that our students *will* learn to "think more clearly."

"In many schools the school day is being lengthened to give pupils a chance to take extra subjects."—*From "Schools in Our Democracy," Office of Education, U.S. Department of Health, Education, and Welfare.*

NCTM

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Minutes of the Annual Business Meeting

Conrad Hilton Hotel, Chicago, Illinois
April 6, 1961

This year a change in procedure of the Annual Business Meeting was instituted. The Annual Business Meeting was scheduled to follow immediately after a general session at which time the President presented a paper entitled "Crystal Gazing." In this paper, President Phillip S. Jones gave a fine survey of the work of the Council during the past year and discussed the projects for the future. Past-President E. H. C. Hildebrandt presided at this general session.

Immediately following this general session, Dr. Phillip S. Jones, President, called the Annual Business Meeting to order at 1:45 P.M.

I. *A motion was made, seconded, and passed to approve the minutes of the meeting of April 22, 1960, as printed in the* Journals.

II. *Report of the 1961 Committee on Nominations and Elections.*

This report was presented by Dr. Oscar F. Schaaf, Chairman. In view of the election returns, the following were officially declared elected:

Vice-President for College: Bruce E. Meserve
Vice-President for Junior High School: Mildred Keiffer
Directors: Max Beberman
William T. Guy, Jr.
Julius H. Hlavaty

At the conclusion of the report, President Jones introduced the newly elected officers and directors.

III. *Report of the Recording Secretary.*

The Board of Directors has held two meetings since the last meeting of the Council in Buffalo in 1960. These meetings were as follows: Tempe, Arizona, December 26–27, 1960; and Chicago, Illinois, April 4–5, 1961. In the brief time allowed, it is impossible to relate all of the work and actions of the Board. Much of the work was routine business, receiving reports from committees, and discussing at length the work of the Council. This report merely intends to inform the membership of the major actions taken.

A. Due to a change in positions, Mr. William H. Glenn resigned as Vice-President. The Board elected Miss Eunice Lewis to fulfill the unexpired term of Mr. Glenn.

B. Dr. M. F. Rosskopf was elected to succeed Dr. W. Warwick Sawyer as Editor of *The Mathematics Student Journal.*

C. In view of the ever-increasing work and the duties of the President, a plan was approved whereby the President of the Council can have released time from his academic duties, up to the equivalent of one academic year, so that he can devote full time to the urgent matters of the Council at least during part of his administration. This type of arrangement has been discussed over a period of several years. Even though this arrangement was made during the presidency of Dr. Jones, the Board insisted that Dr. Jones take advantage of this new provision insofar as possible.

D. The Board approved the recommended changes in the Articles of Incorporation and in the Bylaws and provided the necessary procedure so that the changes in these two documents should be considered at this Annual Business Session of the Council.

E. The Board has approved the following future meetings of the Council:

1961 Summer: Toronto, Canada
1962 Annual (Spring): San Francisco, California
1962 Summer: Madison, Wisconsin
1963 Annual (Spring): Pittsburgh, Pennsylvania
1963 Summer: Eugene, Oregon
1964 Annual (Spring): Miami Beach, Florida

F. One of the most outstanding services rendered by the Council during 1960–61 was the conducting of the Regional Orientation Conferences in Mathematics. This project, under the directorship of Mr. Frank Allen, was supported by a grant from the National Science Foundation. Mr. Allen gave a detailed report to this session of the Board. In the near future, the membership will receive a brochure concerning this program.

G. The membership and the work of the Council continues to grow. The result is more income and greater demands on the Council. In view of these facts, the Board approved a budget of $345,000 for the year 1961–62. A detailed account of this budget will appear in the October issue of the official journals.

H. The Council is ending its first year under the new committee structure as described at the Annual Meeting in Buffalo. Although it has taken time and hard work to bring about the transition from a structure of some thirty committees to a plan of five major committees, the effort has been greatly justified. The change in committee structure does not reduce the number of people who are working on important assignments but merely concentrates the activities into five major groups. Each of the other standing committees of the National Council is now to be regarded as a subcommittee of one of these five major committees. Additional committees as needed will be appointed as subcommittees of these major committees. No attempt will be made to give a digest of the report of these committees now. This is partly due to the fact that some of the committee's actions are merely routine business and partly to the fact that major recommendations have not been completely resolved. From time to time, the membership will be informed of committee proposals as they are adopted. It is sufficient to state that the Council continues to move forward in all areas of its concern.

IV. *Report of the Executive Secretary.*

Mr. M. H. Ahrendt, Executive Secretary, gave the following report, which contains just a few high lights of the administrative and business activities of the Council during the past year.

A. Washington Office:

The full staff of the Washington Office during this school year has consisted of twenty persons. The office again outgrew its floor space and moved to larger quarters in September, 1960. During most of the previous school year the office had been split into two parts, with one office on the fourth floor of the NEA Building and the other on the sixth floor. The office is still divided into two parts, but they are now closer together, being on opposite sides of the corridor on the fourth floor.

Through the co-operation of the Budget Committee and the Board of Directors, the office was again able to purchase this year a number of valuable pieces of labor-saving equipment. This equipment includes an automatic mail opener, an automatic dating stamp for dating mail received, an automatic check endorser, two electric adding machines, one dictating machine, and an electric typewriter.

In July, 1960, a firm of management consultants made a reconnaissance study of the operation of the office. The report of the study characterized the office as a "healthy and well-managed operation." It stated that no problem areas were found in which large-scale studies were needed. The report made certain recommendations for our consideration as the office increases in size. A number of these recommendations have been or are being implemented within the office.

B. Membership:

The membership has continued to grow during the present school year. The present indications are that the total number of members and subscribers will increase by about 20 per cent. It is expected that the membership total on May 1, 1961, will be a little over 37,000. A study of the distribution of subscriptions among our members to the principal journals indicates that 85 per cent of the members take THE MATHEMATICS TEACHER, while 33 per cent take THE ARITHMETIC TEACHER, which means that 18 per cent of the members subscribe to both journals.

C. Circulation of Journals:

The print orders for the issues of each of the three journals for March, 1961, with the per cent of increase over the order for the same issue a year ago, are as follows:

THE MATHEMATICS TEACHER, 30,000 copies, 20 per cent increase

THE ARITHMETIC TEACHER, 17,000 copies, 29 per cent increase

The Mathematics Student Journal, 85,000 copies, 6 per cent increase

It will be noted from the above figures that THE ARITHMETIC TEACHER is growing more rapidly than THE MATHEMATICS TEACHER. This fact is true both among the membership subscriptions and the institutional subscriptions, but the growth is more rapid among the latter.

D. Financial Condition:

The financial condition of the Council continues to be very good, although it is relatively a little less favorable than it was a year ago. This is probably a result of the fact that our membership is not growing as rapidly now as it was growing a year ago. It seems likely that our total receipts during this fiscal year will be approximately $373,000. The receipts will probably exceed the expenditures by about $46,000, thus increasing our total resources by that amount. The ratio of total cash resources to total expenses has remained quite stable over a period of years. Total cash resources are equal to

about two-thirds of the total expenses for a year, giving us a "cushion" for emergencies.

The incomes from memberships and the sale of yearbooks have both increased during the year in close step with the increase in membership. However, the increase in the sale of pamphlets and supplementary publications has been less. We need to find a way to restore the vitality of our sales program for supplementary publications.

E. Publications:

The chief new publication for this year is, of course, the 26th Yearbook, *Evaluation in Mathematics*. The book is off the press and copies will be on sale at this meeting.

The following pamphlets have come off the press since the Annual Meeting in Buffalo a year ago: *The High School Mathematics Library*, 29,776 copies; *The Elementary and Junior High School Mathematics Library*, 19,200 copies; *The Number Story*, 7,000 copies; *Membership Directory*, 28,150 copies; *Careers in Mathematics*, 500,000 copies; *Vectors in Three Dimensional Geometry*, 5,000 copies; *Secret Codes, Remainder Arithmetic, and Matrices*, 7,000 copies; *Some Ideas About Number Theory*, 5,000 copies.

The following publications have been reprinted during the past year: *23rd Yearbook*, 10,000 copies; *21st Yearbook*, 3,000 copies; *Number Stories of Long Ago*, 7,700 copies; *How to Use Field Trips in Mathematics*, 6,000 copies; *How to Use Films and Filmstrips in Mathematics Classes*, 6,000 copies; *A Portrait of 2*, 5,000 copies; *Designing the Mathematics Classroom*, 3,000 copies; *How to Use Your Library in Mathematics*, 6,000 copies; *Education in Mathematics for the Slow Learner*, 6,000 copies.

F. Free Materials Mailed to Members:

The convention program booklets and *Membership Directory* are of course routinely sent free to members. About 13,800 copies of *The Elementary and Junior High School Mathematics Library* and 25,700 copies of *The High School Mathematics Library* were mailed free to individual members and institutional subscribers. The *TV Lesson Schedule* for the second semester of "Continental Classroom" was mailed to members with the cooperation of Learning Resources Institute, which provided the printed copies. The NCTM contributed its mailing list, the supplies, postage, and labor. About 25,000 copies of *Careers in Mathematics* were mailed to individual members a few weeks ago. In addition, members received numerous professional materials, some of a valuable nature, through the release of the membership list to other organizations.

G. Careers in Mathematics:

It is probably appropriate to make special mention of the pamphlet *Careers in Mathematics*, since it marks the completion of a project of much significance to the Council and to teachers and students of mathematics. This project represents the outgrowth of two separate projects initiated a number of years ago by the Council and the National Academy of Sciences-National Research Council. Without the dedicated and persistent help of numerous individuals from the latter organization, the pamphlet could not have been developed. This fine pamphlet contains "profiles" of eight young mathematicians, picturing the kind of work they do and the kind of persons they are. The pamphlet also contains an authoritative section on guidance and a list of addresses of persons and institutions to which the interested student may write for further information.

The Washington Office was able to assist with the project by taking complete responsibility for the details of production and distribution, and also by assisting with the editing, helping with the final writing, assisting with the design and art, and working out the distribution plan.

The funds which the National Academy of Sciences-National Research Council had been able to raise provided for the printing of a half-million copies. Approximately 77,000 of these were mailed singly to teachers of mathematics throughout the United States. Requests for quantities for distribution to students are now being received by your Washington Office. It has been necessary for us to devise a quota system for assigning copies so that the needs of students in all parts of the United States may be met. At the present time about four-fifths of the free quantities have been distributed. Through authorization of our Publications Committee, this pamphlet will be retained indefinitely, after the free distribution has been completed, as a sales item which teachers may purchase in any quantity.

H. Other Services:

The Washington Office has been able to assist in various ways with a number of other important and significant projects during the year.

In the fall of 1960, we made, at the request of a committee of the Congress, a study of the opinions of teachers of mathematics on the operation of the National Defense Education Act of 1958.

Most of the work involved in gathering information for the preparation of proposed amendments to the Articles of Incorporation and much of the work of arranging the amendments to the Bylaws

was done in the office.

The office was able to assist considerably with the administration of our series of eight Regional Orientation Conferences in Mathematics by handling the funds; supervising the printing and mailing of reports, stationery, and the like; and by answering correspondence.

The office also assisted with numerous other projects both of a special and a routine nature. The past year has undoubtedly been the busiest one which the Washington Office has experienced.

V. *Proposed Amendments to the Articles of Incorporation and Bylaws.*

SECRETARY'S NOTE: The proposed amendments to these two documents were prepared by a committee appointed by the Board and composed of M. H. Ahrendt, Houston T. Karnes, Philip Peak, and Milton W. Beckman, Chairman. In addition, the services of legal counsel were employed. The committee worked independently and on two occasions worked with the Board. The final drafts of these two documents as approved by the Board were printed and distributed to the membership of the Council under the provisions of the existing Articles of Incorporation and Bylaws. Copies were also available and distributed to this session of the Annual Meeting.

A. Articles of Incorporation:

The Recording Secretary presented the proposed amendments to the membership, whereupon a motion was made and seconded to adopt the proposed amendments. Following a brief discussion, the question was called for, whereupon President Jones put the motion to a vote. The motion was unanimously carried.

B. Bylaws:

The Recording Secretary presented the proposed amendments to the membership, whereupon a motion was made and seconded to adopt the proposed amendments. Following a lengthy discussion, the question was called for, whereupon President Jones put the motion to a vote. The motion was unanimously carried.

SECRETARY'S NOTE: Copies of the newly adopted Articles of Incorporation and Bylaws are to be found beginning on page 469 of this issue.

VI. *Qualifications for Presidential Nominees for 1962.*

Motion: A motion was made, seconded, and passed that the Nominating Committee for 1962 be guided by the old Bylaws with regard to qualifications for a presidential nominee.

VII. *Resolutions.*

A committee appointed by President Jones and composed of J. Houston Banks, Harold P. Fawcett, and Burton W. Jones presented the following resolutions.

A. The "Continental Classroom":

The National Council of Teachers of Mathematics wishes to express its appreciation to those organizations responsible for and supporting the "Continental Classroom" in view of its unique contribution to the improvement of mathematics.

The recording secretary was hereby instructed to convey the appreciation of the Council to these organizations.

SECRETARY'S NOTE: These organizations are as follows: sponsors—The Conference Board of the Mathematical Sciences, Learning Resources Institute, National Broadcasting Company; donors—Bell Telephone System, E. I. Du Pont de Nemours & Company, General Foods Fund, IBM Corporation, Radio Corporation of America, The Ford Foundation, Union Carbide Corporation, United States Steel.

Motion: A motion was made, seconded, and passed to adopt this resolution.

B. Appreciation:

The National Council of Teachers of Mathematics wishes to express its gratitude to The Illinois Council of Teachers of Mathematics, The Chicago Elementary Teachers Mathematics Club, Men's Mathematics Club of Chicago and Metropolitan Area, and Women's Mathematics Club of Chicago and Suburbs for a tremendous job well done.

To Mr. Hobert Sistler, Chairman of the General Planning Committee, and to his twenty local committees of 253 members, the Council extends its sincere thanks for the staggering amount of work done in planning and preparing for this great convention of some ninety separate meetings.

The management of the Conrad Hilton is also to be commended for its friendly and efficient service in caring for the convention needs of some 3,500 persons.

The recording secretary is hereby instructed to convey to the appropriate persons the spirit of this resolution.

VIII. *Announcement.*

Professor J. Sutherland Frame, who has recently been appointed by the Conference Board of the Mathematical Sciences to direct a study of the design of facilities for mathematics in colleges and schools, made the following announcement in substance:

In connection with his new appointment, Professor Frame called upon the membership to convey to him any ideas and suggestions which might be helpful in this undertaking.

IX. *There being no further business, the meeting was duly adjourned at* 3:00 P.M.

Respectfully submitted,
HOUSTON T. KARNES,
Recording Secretary

Annual financial report

by M. H. Ahrendt, Executive Secretary, NCTM, Washington, D.C

The report that follows gives a summary of the receipts and expenses of The National Council of Teachers of Mathematics for the fiscal year ending May 31, 1961. The increases in both receipts and expenditures were approximately proportional to our increase in membership during the year.

Readers may be especially interested in the following items. (1) For the first time in our history our total sale of yearbooks and pamphlets exceeded $100,000. (2) A significantly large expenditure was made for the printing and distribution of free materials to members. (3) The largest per cent of increase during the year in any item was in the sale of advertising space in the journals, which showed an increase of 50 per cent. This increase is probably a result of the fact that, through the stimulus provided by the NDEA, many new companies are entering the field of teaching aids for mathematics.

Receipts and expenditures of The National Council of Teachers of Mathematics for the fiscal year, June 1, 1960-May 31, 1961

June 1, 1960—Total Cash Resources		$171,793.76
RECEIPTS		
Memberships with subscriptions to THE MATHEMATICS TEACHER	$ 95,215.42	
Memberships with subscriptions to THE ARITHMETIC TEACHER	35,971.05	
Institutional subscriptions to THE MATHEMATICS TEACHER	39,326.50	
Institutional subscriptions to THE ARITHMETIC TEACHER	46,686.85	
Subscriptions to THE MATHEMATICS STUDENT JOURNAL	19,237.30	
Sale of advertising space in THE MATHEMATICS TEACHER	19,773.08	
Sale of advertising space in THE ARITHMETIC TEACHER	4,567.45	
Rental of membership list	5,868.24	
Interest on investments	3,387.49	
Net profit from conventions	3,927.20	
Miscellaneous	746.09	
Sale of publications		
Yearbooks	67,249.17	
Miscellaneous	33,331.28	
Total receipts	$375,287.12	
EXPENDITURES		
Washington Office	$121,494.72	
President's Office	4,989.76	
Vice-Presidents' Office expenses	781.44	
THE MATHEMATICS TEACHER	52,167.23	
THE ARITHMETIC TEACHER	31,229.17	
THE MATHEMATICS STUDENT JOURNAL	11,133.45	
Committee secretarial expenses	989.99	
Committee travel	4,476.39	
Subcommittee work	6,453.54	
Travel by Board members and Program Planning Committee	6,133.81	
Management survey of Washington office	750.00	
Preparation and printing of yearbooks	31,001.81	
Preparation and printing of supplementary publications	13,395.69	
Printing and distributing free pamphlets	7,482.36	
Printing and distributing 1960 Membership Directory	11,573.03	
Printing and distributing Amendments to Bylaws	1,941.51	
Storage and shipment of publications, miscellaneous	4,306.62	
Total expenditures	$310,300.52	
Increase in Cash Resources		$ 64,986.60
May 31, 1961—Total Cash Resources		$236,780.36

The 1961 budget

The Committee on Financial Policies, Budget, and Auditing for 1961 was composed of Burton W. Jones, Bruce E. Meserve, Vernon H. Price, Chairman, and M. H. Ahrendt, ex officio.

The Committee met at the Washington Office, January 24–25, 1961. During this period, the Committee had the fine services of the Executive Secretary, M. H. Ahrendt, and the availability of all the important records in the Central Office.

On the basis of the work done during this meeting, a proposed budget was prepared and submitted by mail to the members of the Board on February 15, 1961. At the Chicago Meeting, the Budget was presented by the Chairman and discussed thoroughly by the Board. The Budget as adopted by the Board, which includes a few modifications of the Budget as prepared by the Committee, is submitted below:

RECEIPTS		
Memberships (30,000 @ $5.00)	$150,000	
Memberships (5,000 @ $1.50)	7,500	
MSJ Subscriptions (75,000 @ $.30)	22,500	
Memberships (for second journal: 5,100 @ $3.00)	15,300	
Subscriptions, journals (7,500 @ $7.00)	52,500	
Subscriptions, journals (7,500 @ $6.75)	50,625	
Advertising in journals	20,200	
Interest on U.S. Treasury bonds	375	
Interest on savings accounts	6,000	
From reserves	20,000	
Total receipts		$345,000
EXPENDITURES		
Washington Office		
Executive Secretary		
Salary	$ 12,000	
Professional assistant		
Salary	8,000	
Secretarial, editorial, and clerical help		
Salaries	87,000	
General		
General office expenses	31,000	
Travel	4,000	
Special benefits (Social Security, hospitalization, and retirement)	10,000	
Total		$152,000
President		
Release of President's time	$ 7,000	
Office expenses	1,500	
Secretary	5,000	
Travel	10,000	
Total		$ 23,500
Vice-Presidents		
Office expenses	$ 1,000	
Total		$ 1,000
THE MATHEMATICS TEACHER		
Printing and mailing	$ 50,000	
Edit, Inc.	3,600	
Secretary	1,500	
Office expenses and equipment	800	
Travel	1,000	
Total		$ 56,900

THE ARITHMETIC TEACHER		
Printing and mailing	$ 27,500	
Edit, Inc.	3,300	
Secretary	2,000	
Office expenses and equipment	800	
Travel	1,000	
Promotion of THE ARITHMETIC TEACHER	1,000	
Total		$ 35,600
THE MATHEMATICS STUDENT JOURNAL		
Printing and mailing	$ 15,000	
Office expenses and equipment	500	
Secretary	1,000	
Travel	1,000	
Total		$ 17,500
Committee Expenses		
Executive Committee	$ 24,000	
Professional Standards Committee	2,000	
Professional Relations Committee	14,000	
Publications Committee	4,500	
Plans and Proposals Committee	2,500	
Total		$ 47,000
Miscellaneous Expenses		
Travel: Vice-Presidents, Directors, and Recording Secretary	$ 7,500	
Subsidization of conventions	4,000	
Total		$ 11,500
Total expenditures		$345,000

Articles of Incorporation

(As amended and adopted at the Annual Business Meeting, April 6, 1961)

1. The name of such corporation is The National Council of Teachers of Mathematics (Incorporated)

2. The object for which it is formed is to assist in promoting the interests of mathematics in America, especially in the elementary and secondary fields, by holding meetings for the presentation and discussion of papers, by conducting investigations for the purpose of improving the teaching of mathematics, by the publication of papers, journals, books, and reports, thus to vitalize and co-ordinate the work of many local organizations of teachers of mathematics and to bring the interests of mathematics to the attention and consideration of the educational world.

3. The management of the corporation shall be vested in a board of not less than three directors, or such greater number of directors as may be provided for in the bylaws of the corporation, as amended from time to time.

4. The corporation shall be exclusively a charitable, scientific, and educational corporation within the meaning of Section 501(c) (3) of the Internal Revenue Code of 1954, as currently in effect or as hereafter amended from time to time, provided, however, that the corporation shall not operate a post-secondary educational institution or vocational school, nor shall it be authorized to receive any child for care or placement apart from its own parent or guardian.

5. The corporation is not organized for profit or pecuniary benefit. It is a membership corporation without stock or stockholders. No part of the revenue or income of the corporation shall inure, either directly or indirectly, to the benefit of any member, officer, employee or any private individual or to the benefit of any corporation or organization, any part of the net earnings of which inure to the benefit of any private individual, provided, however, that this shall not prevent payment of reasonable compensation for services actually rendered to or for the corporation in effecting its purposes.

6. The corporation shall not, directly or indirectly:

a) Devote any substantial part of its activities to the carrying on of propa-

ganda or otherwise attempting to influence legislation, or participate or intervene, by the publication or distribution of statements or otherwise, in any political campaign on behalf of any candidate for public office;

b) Make any accumulation of income unreasonable in amount or duration or invest income in any manner to jeopardize the fulfillment of the objects of the corporation;

c) Engage in any activity affecting the corporation's right to full tax exemption as a charitable, scientific or educational organization under Federal laws or the laws of states or local governmental bodies;

d) Make any loan to any of its officers or directors;

nor shall it, directly or indirectly, devolve any pecuniary or economic benefit upon any member, sponsor, donor, creator, director, officer, or employee by reason or virtue of, or through:

e) The loan of any part of its income or property to any such person or organization without the receipt of adequate security and a reasonable rate of interest; or

f) The purchase of any property from any such person or organization for more than adequate consideration in money or money's worth or the sale to any such person or organization for less than adequate consideration in money or money's worth or any other transaction with any such person or organization which might result in the diversion of the income or property of the corporation.

7. If, at any time, the corporation shall cease to carry out the purposes as herein stated, all assets and property held by it, whether in trust or otherwise, shall, after the payment of its liabilities, be paid over to an organization, selected by the final Board of Directors of the corporation, which has similar purposes and has established its tax-exempt status under Section 501(c)(3) of the Internal Revenue Code of 1954 as now enacted or as it may hereafter be amended, and such assets and property shall be applied exclusively for such charitable, scientific, and educational programs.

8. The following persons are hereby selected as the Directors to control and manage said corporation for the first year of its corporate existence, viz.:

Charles M. Austin
333 North Oak Park Avenue
Oak Park, Illinois

H. E. Slaught
5548 Kenwood Avenue
Chicago, Illinois

Lillis Price
631 Lake Avenue
Wilmette, Illinois

9. The location is in the city of Chicago in the county of Cook in the State of Illinois, and the post office address of its business office is at No. 5548 Kenwood Avenue, in the said city of Chicago, Illinois.

Bylaws

(*As amended April 6, 1961*)

Article I
Name, Purpose, Affiliation, and Corporate Seal

A. Name. This organization shall be known as The National Council of Teachers of Mathematics (Incorporated).

B. Purpose. Its object shall be to assist in promoting the interests of mathematics in America, especially in the elementary and secondary fields, by holding meetings for the presentation and discussion of papers, by conducting investigations for the purpose of improving the teaching of mathematics, by the publication of papers, journals, books, and reports, thus to vitalize and co-ordinate the work of many local organizations of teachers of mathematics and to bring the interests of mathematics to the attention and consideration of the educational world.

C. Affiliation. This organization shall be a Department of the National Education Association. The Council shall encourage its members to become members of the National Education Association.

D. Corporate Seal. The corporate seal of the Council shall have inscribed thereon the name of the organization and the words "Corporate Seal—Illinois."

Article II
Membership

A. Any person who is interested in mathematics shall be eligible for membership in the Council.

B. The Board of Directors may establish various types of membership in the Council, determine the fee for each, and designate the publications included. The Mathematics Teacher and The Arithmetic Teacher shall be the official journals.

Article III
Officers, Board of Directors, Committees

Section I. Officers

A. Elective Officers. The elective officers of the Council shall be the president, the president-elect, and the four vice-presidents.
   1. Qualifications
      a) A nominee for president-elect must have been a member of the Council continuously during the preceding five years and have participated in the activities of the Council.
      b) A nominee for vice-president must have been a member of the Council continuously during the preceding three years, and have participated in the activities of the Council.
      c) There shall be a vice-president to represent each of the following fields: Elementary School, Junior High School, Senior High School, and College.
   2. Election
      a) The elective officers shall be elected by members of the Council by mail ballot.
      b) Two vice-presidents shall be elected each year.
   3. Tenure
      a) The president shall serve for two years or until his successor is installed and shall not be eligible for re-election.
      b) The president-elect shall serve for one year and then be installed as president.
      c) Each vice-president shall serve for two years or until his successor is installed and shall not be eligible for re-election.
   4. Compensation. Compensation, if any, for officers shall be fixed by the Board of Directors.
   5. Duties
      a) President. The president shall be the chief officer of the Council. He shall preside at all meetings of the Board of Directors and at the annual business meeting of the Council. He shall have the usual duties pertaining to his office and such other duties as may from time to time be assigned him by the Board of Directors. He shall appoint all committees, subject to the approval of the Board of Directors.
      b) President-elect. The president-elect shall be a nonvoting member of the Board of Directors.
      c) Vice-presidents
         (1) The vice-presidents shall be ranked first according to seniority in the office and then according to the number of votes received at the time of election.
         (2) In the event of the president's inability to serve or the inability of the president-elect to assume office as scheduled, the vice-presidents shall assume in the above order the responsibilities of the presidency.

B. Appointive Officers. The appointive officers of the Council shall be the executive secretary, the recording secretary, and the editors of the official journals.
   1. Appointment. The executive secretary, the recording secretary, and the editors of the official journals shall be appointed by the Board of Directors.
   2. Tenure. The executive secretary shall be appointed for a term of three years; he may be reappointed. The editors of the official journals shall be appointed for a term of three years; they may be reappointed but shall not serve for more than two consecutive terms. The recording secretary shall be appointed for a term of one year; he may be reappointed.
   3. Compensation. Compensation, if any, for appointive officers shall be fixed by the Board of Directors.
   4. Duties
      a) Executive Secretary. The executive secretary shall, working under the direction of the president, endeavor to carry out the policies formulated by the Council, the Board of Directors, and the Executive Committee. He shall serve as treasurer and business manager of the Council.
      b) Recording Secretary. The recording secretary shall keep a record of the proceedings of all business meetings of the Council and of the Board of Directors.
      c) Editors. Each editor shall be responsible for his journal in all phases except that of business management.

Section II. Directors

A. Number. There shall be nine directors.
B. Qualifications. A nominee for director must have been a member of the Council continuously during the preceding three years.
C. Election. The directors shall be elected by members of the Council by mail ballot according to a geographic plan determined by the Board. Three directors shall be elected each year.
D. Tenure. The directors shall serve for a term of three years or until their successors are installed. They shall be eligible for re-election but shall not serve for more than two consecutive terms.

Section III. Board of Directors

A. The voting members of the Board of Directors shall consist of the following: the president, the vice-presidents, and the directors. The executive secretary, the recording secretary, the editors of the official journals, and the president-elect shall be nonvoting ex-officio members.
B. Duties. The Board of Directors shall have general supervision of the activities of the

Council, supervise the expenditure of funds, fix salaries and bonds of officers, fill vacancies ad interim in any office, appoint editors of yearbooks, and make all appointments not specifically delegated to the president.

Section IV. Committees

A. The Executive Committee shall consist of the president as chairman, at least two voting members of the Board of Directors, and the executive secretary as a nonvoting ex-officio member.
   1. Appointment. The president shall appoint the members of the Executive Committee, subject to the approval of the Board. They shall serve for one year.
   2. Duties. This committee shall act for the Board of Directors when the Board is not in session and shall report all such actions at the next meeting of the Board. It shall also act in an advisory capacity to the president.

B. The president shall, subject to the approval of the Board of Directors, appoint whatever standing and special committees are deemed necessary to carry out the purposes of the Council.

Article IV
Meetings, Quorums, Rules of Order

A. The Council shall hold at least one convention per year, which shall be called the Annual Meeting, at such time and place as the Board may direct. There shall be a business meeting of the Council in conjunction with the Annual Meeting.

B. Special business meetings of the Council shall be called by the president under the authorization of a majority of the members of the Board of Directors.

C. The Board of Directors shall hold a meeting at the time of the Annual Meeting. Additional meetings of the Board shall be held at the call of the president or on the written request of two-thirds of the voting members of of the Board.

D. Notice of a business meeting of the Council shall be given to the members by the executive secretary at least thirty days prior to the date set for said meeting. Notice of any meeting of the Board other than regular meetings herein provided shall be given to each member of the Board at least fifteen days prior to the date set therefor.

E. At any business meeting of the Council a quorum shall consist of seventy-five members.

F. A majority of the voting members of the Board of Directors shall constitute a quorum for the transaction of business.

G. An announcement inviting members of the Council to suggest nominees for elective offices shall be published in each of the official journals approximately eleven months before the annual meeting at which the election result is to be announced. There shall be placed in nomination at least two persons for each elective office, and the Board shall instruct the executive secretary to mail ballots to all members at least thirty days before the Annual Meeting. A date for the closing of the balloting shall appear on the printed ballot.

H. *Robert's Rules of Order, Revised* shall govern the conduct of all business meetings of the Executive Committee, Board of Directors, and the Council.

Article V
Affiliated Groups

A. Any organized group of teachers of mathematics or any group of individuals preparing to become teachers of mathematics may petition the Board of Directors to become affiliated with the Council. The Board shall specify the conditions under which such affiliation may take place.

B. An affiliated group shall be entitled to send a delegate to the Delegate Assembly.

C. The Delegate Assembly shall be composed of representatives from affiliated groups. The Assembly shall hold a meeting at the time of the Annual Meeting of the Council, and shall serve as a forum which may make recommendations to the Board of Directors concerning activities and policies of the Council.

Article VI
Amendments

A. Amendments to these Bylaws may be made at any annual business meeting of the Council or special meeting called for that purpose, providing that due notice concerning such amendments shall have been printed in the official journals or mailed to each member at least thirty days before the date of such meeting.

"Research has been initiated to improve teacher training through television, to develop self-learning devices, and to prepare and test new materials in language, science, mathematics, history, and engineering."—*From "Schools in Our Democracy," Office of Education, U.S. Department of Health, Education, and Welfare.*

Committees and Representatives (1961-1962)

SECRETARY OF THE BOARD
Houston T. Karnes, Baton Rouge, La.

EXECUTIVE COMMITTEE
Phillip S. Jones, Ann Arbor, Mich., Chairman, (1962)
Philip Peak, Bloomington, Ind. (1962)
Henry Van Engen, Madison, Wis. (1962)
Myrl H. Ahrendt, Washington, D.C. (ex officio)

Nominations and Elections Committee
(To begin and end terms in September of the years stated)
W. Eugene Ferguson, Newton, Mass., Chairman until Sept. 1961, member until Sept. 1962
Agnes Herbert, Baltimore, Md., Chairman, Sept. 1961–Sept. 1962, member until 1963
Mike Donahoe, Carmel, Calif. (1963)
Harold Fawcett, Columbus, Ohio (1962)
Catherine A. V. Lyons, Pittsburgh, Penn. (1964)
Irene Sauble, Detroit, Mich. (1963)
Lottchen Hunter, Wichita, Kan. (1962)
Lenore John, Chicago, Ill. (1961)
W. C. Lowry, Charlottesville, Va. (1961)
Nick L. Massey, Seattle, Wash. (1963)
Lyman C. Peck, Oxford, Ohio (1964)
Oscar Schaaf, Eugene, Ore. (1961)
Max Sobel, Fair Lawn, N.J. (1961)

Committee on Financial Policies, Budget, Auditing
Bruce Meserve, Montclair, N.J., Chairman (1962)
Julius Hlavaty, New York, N.Y. (1964)
Burton W. Jones, Boulder, Col. (1963)
Myrl H. Ahrendt, Washington, D.C. (ex officio)

Place of Meeting Committee
Lauren Woodby, Mt. Pleasant, Mich., Chairman (1962)
Arnold M. Chandler, Madison, Wis. (1964)
Mary McDermott, Concord, Calif. (1964)
Harry L. Phillips, Charleston, W. Va. (1964)
Ella S. Porter, Houston, Tex. (1962)
Myrl H. Ahrendt, Washington, D.C. (ex officio)

Committee on the Analysis of Special Programs
Philip Peak, Bloomington, Ind., Chairman (1962)
Alice M. Hach, Ann Arbor, Mich. (1964)
Kenneth B. Henderson, Urbana, Ill. (1963)
Z. L. Loflin, Lafayette, La. (1962)
Myron Rosskopf, New York, N.Y. (1963)

Research Committee
John Kinsella, New York, N.Y., Chairman
Kenneth Brown, Washington, D.C.
Howard F. Fehr, New York, N.Y.
Sheldon Myers, Princeton, N.J.
J. Fred Weaver, Boston, Mass.

Computer Project Committee
W. Eugene Ferguson, Newton, Mass., Chairman
Myrl H. Ahrendt, Washington, D.C.
Leon Ellsworth, New York, N.Y.
Kenneth Henderson, Urbana, Ill.
Burton W. Jones, Boulder, Col.
Phillip S. Jones, Ann Arbor, Mich.
Harley Tillitt, China Lake, Calif.

PROFESSIONAL STATUS AND STANDARDS COMMITTEE
John R. Mayor, Washington, D.C., Chairman (1962)
Frank B. Allen, La Grange, Ill. (1963)
John A. Brown, Newark, Del. (1964)
W. T. Guy, Jr., Austin, Tex. (1962)
Mildred Keiffer, Cincinnati, Ohio (1963)

Membership Committee
Mary C. Rogers, Westfield, N.J., Chairman
Pearl Bond, Beaumont, Tex.
Janet Height, Wakefield, Mass.
Lucille Houston, Racine, Wis.
Harold J. Hunt, Seattle, Wash.
Florence E. Ingham, Bartlesville, Okla.
Faith Novinger, Washington, D.C.

PROFESSIONAL RELATIONS COMMITTEE
Irvin Brune, Cedar Falls, Iowa, Chairman (1962)
J. Houston Banks, Nashville, Tenn. (1962)
Veryl Schult, Washington, D.C. (1963)
Joseph F. Senta, Minneapolis Minn. (1963)
Howard Fehr, New York, N.Y. (1964)

Affiliated Groups Committee
Eugene Smith, Detroit, Mich., Chairman (1963)
Vincent Brunner, Milwaukee, Wis. (1964)
Mary Louise Fisher, Joliet, Ill. (1964)
Mary Rogers, Westfield, N.J. (1962)
Kenneth Skeen, Concord, Calif. (1964)
Keene C. Van Orden, San Angelo, Tex. (1963)
Annie John Williams, Durham, N.C. (1962)

International Mathematics Education Committee
E. H. C. Hildebrandt, Evanston, Ill., Chairman (1963)
Howard Fehr, New York, N.Y. (1963)
Burton W. Jones, Boulder, Col. (1963)

Alfred L. Putnam, Chicago, Ill. (1962)
Ernest R. Ranucci, Newark, N.J. (1962)
Veryl Schult, Washington, D.C. (1962)

Committee on Relations with Industry
Marie S. Wilcox, Indianapolis, Ind., Chairman (1962)
Clifford Bell, Los Angeles, Calif.
George Rietz, New York, N.Y.
Edward Spacapan, Arlington Heights, Ill.

NCTM Representatives

AAAS Co-operative Committee
H. Vernon Price, Iowa City, Iowa (1964)

Conference Board of the Mathematical Sciences
Burton W. Jones, Boulder, Col. (1962)
John R. Mayor, Washington, D.C.

U.S. Commission on Mathematical Instruction
E. H. C. Hildebrandt, Evanston, Ill. (1963)
Burton W. Jones, Boulder, Col. (1964)

AAAS Council
H. Glenn Ayre, Macomb, Ill. (1963)
James H. Zant, Stillwater, Okla. (1962)

Council on Co-operation in Teacher Education
Veryl Schult, Washington, D.C. (1962)

Delegates to TEPS Meeting
Lee E. Boyer, Harrisburg, Pa. (1961)
Earle F. Myers, New Brighton, Pa. (1961)

NEA Representative Assembly
Julius H. Hlavaty, New York, N.Y. (1961)

Publications Committee
Donovan A. Johnson, Minneapolis, Minn., Chairman (1963)
Paul Johnson, Los Angeles, Calif. (1963)
Ben A. Sueltz, Cortland, N.Y. (1962)
Henry Van Engen, Madison, Wis. (1962)
Lenore John, Chicago, Ill. (1964)

Ex-officio nonvoting members
Myrl H. Ahrendt, Washington, D.C.
E. Glenadine Gibb, Editor, The Arithmetic Teacher
Robert E. Pingry, Editor, The Mathematics Teacher
Myron F. Rosskopf, Editor, The Mathematics Student Journal

Supplementary Publications Committee
Kenneth Henderson, Urbana, Ill., Chairman (1962)
Marguerite Brydegaard, San Diego, Calif. (1962)
M. Albert Linton, Philadelphia, Pa. (1964)
Eugene Nichols, Tallahassee, Fla. (1963)
Helen Schneider, Oak Park, Ill. (1963)
Daniel W. Snader, Washington, D.C. (1964)

Yearbook Planning Committee
Ben A. Sueltz, Cortland, N.Y., Chairman (1963)
J. Houston Banks, Nashville, Tenn. (1962)
Edwin C. Douglas, Watertown, Conn. (1964)
Joseph J. Stipanowich, Macomb, Ill. (1964)

Editorial Board, The Mathematics Teacher
Robert E. Pingry, Urbana, Ill., Editor (1962)
Jackson B. Adkins, Exeter, N.H.
Mildred Keiffer, Cincinnati, Ohio
Daniel B. Lloyd, Washington, D.C.
Z. L. Loflin, Lafayette, La.
Ernest Ranucci, Union, N.J.

Editorial Board, The Arithmetic Teacher
E. Glenadine Gibb, Cedar Falls, Iowa, Editor (1963)
E. W. Hamilton, Cedar Falls, Iowa
Edwina Deans, Washington, D.C.
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J. Fred Weaver, Boston, Mass.
Marguerite Brydegaard, San Diego, Calif.
John R. Clark, New Hope, Penn.
Vincent J. Glennon, Syracuse, N.Y.

Editorial Board, The Mathematics Student Journal
Myron F. Rosskopf, New York, N.Y., Editor (1964)
Stephen Willoughby, Madison, Wis.
Mannis Charosh, New York, N.Y.

Editorial Committee, *Twenty-seventh Yearbook* (Mathematics for the Talented)
Julius H. Hlavaty, New York, N.Y., Chairman
Albert Blank, New York, N.Y.
Vincent J. Glennon, Syracuse, N.Y.
Joseph N. Payne, Ann Arbor, Mich.
Richard Pieters, Andover, Mass.
Harry Ruderman, New York, N.Y.
Henry Syer, Kent, Conn.

Plans and Proposals Committee
Robert E. K. Rourke, Kent, Conn., Chairman (1962)
Roy Dubisch, Fresno, Calif. (1962)
William Matson, Portland, Ore. (1963)
Henry Van Engen, Madison, Wis. (1964)

"The school year is being extended beyond one hundred eighty days in many districts.—*From "Schools in Our Democracy," Office of Education, U.S. Department of Health, Education, and Welfare.*

Report of the Membership Committee

Mary C. Rogers, Chairman, Membership Committee,
Edison Junior High School, Westfield, New Jersey

You who were present at the Thirty-eighth Annual Meeting of The National Council of Teachers of Mathematics in April, 1960, will remember the remarkable membership growth which we were able to report and the enthusiasm with which it was received throughout the convention. The combined efforts of the personnel of a great organization working toward a common goal had brought about results far beyond expectations.

Portions of this report were later published in editions of THE MATHEMATICS TEACHER and THE ARITHMETIC TEACHER and the expanded goal of 40,000 members to be reached by April 1961 announced.

What progress have we made toward the realization of this goal? May we present a few outstanding facts and leave it to you to draw your own conclusions.

The May 1, 1960, total membership was 31,194, or 194 more than the estimate for that date.

The April 6, 1961, reported total—based on the February 1, 1961, count, was 34,491—5509 less than the 40,000 hoped for that date.

By May 1, 1961, nearly *3000 additional memberships* had been reported, bringing the new total to 37,224, slightly more than had been originally anticipated for this date, but nearly 3000 less than the newly accepted goal of 40,000. This total, however, is slightly more than 93% of total goal.

Other interesting data show:

14 states and territories having reached their shares in the 40,000 goal or gone beyond.

10 states and territories with membership achievements of 90–99%.

7 states and territories with achievements of 85–89%.

1 of the seven regions has already reached 120% of new goals.

5 of the remaining 6 regions have reached 90–93% achievements.

The *total membership growth for 1960–61 has been 6030,* as compared with the 7350 increase for the previous year. This is a little disappointing, but we still think the record is quite commendable when the circumstances are known.

Since we have grown to be such a large organization, the Board of Directors has found it most difficult to transact all necessary business through direct reports from the many committees heretofore reporting to them directly. This has resulted in a complete reorganization of committees and committee services. You will recall reading about this in THE MATHEMATICS TEACHER, October, 1960 issue.

There are now five major committees supplemented by such other committees as the Board deems essential. Some of the former committees have been discontinued; others will continue their services as subcommittees of the five major committees.

At the Christmas Meeting in Tempe it was ruled that: "The Membership Committee should be continued as a subcommittee of the Committee on Professional Standards and should proceed as customary in the past until such time as further growth of the organization or change in organizational procedures make a revision of plans and procedures necessary."

As soon as reactivation of the Membership Committee services had thus become official, your Chairman contacted Dr.

John R. Mayor, Chairman of the Committee on Professional Standards, for instructions. Dr. Mayor immediately wrote expressing in reply his pleasure that the Membership Committee would be continued and offering whatever assistance we desired as a subcommittee under his direction.

The Membership Committee has been reduced in size from ten persons to seven persons. This has made necessary a regrouping of the States as to Regions and Membership Committee leadership in these Regions.

As a part of Membership reactivation, your former Membership Committee was reappointed by Dr. Jones and by the Board of Directors and was given their expanded assignments by the Chairman. Furnished with the latest membership count and analyses, they have been instructed to resume the procedures found so effective in the past—with added emphasis on:

1 An *expanded program of publicity* and an increased program of *judicious advertising* which will point out the purposes of the Council together with the *many benefits* to be gained from membership.
2 *Greater attention given to renewals* as well as to solicitation of new memberships.
3 *Closer co-operation and articulation of effort* between the Membership Committee, the Affiliated Groups, and the State Representatives.

At its meeting in Chicago, the Membership Committee set up a definite schedule of specific procedures whereby it hopes to perform a more effective service to you and to the Council. We welcome any suggestions you may have for us in bringing about this improvement. You will be hearing from us rather regularly—in line with our established schedule.

In the tables that follow you will find rather complete analyses of current membership status by States and by Regions. We believe that you will find these interesting and helpful. You will see that progress this year—although still very good—has suffered a slight setback when compared to last year's remarkable growth. We are convinced you are determined to correct this condition and make next year another banner year. Your enthusiasm and eager interest at the Chicago convention are indicative of this strong intention.

If we make another 7000 membership growth, or even 6000 growth, we will be well beyond our 40,000 goal. A reactivated Committee aided by your fine co-operation and great helpfulness will easily bring about this high objective.

Your Membership Committee thanks you most sincerely for your assistance at all times, and extends to you its best wishes for your greatest success in all of your professional endeavors.

PEARL BOND, *Beaumont, Tex.*
JANET HEIGHT, *Wakefield, Mass.*
LUCILLE HOUSTON, *Racine, Wis.*
HAROLD J. HUNT, *Seattle, Wash.*
FLORENCE INGHAM, *Bartlesville, Okla.*
FAITH NOVINGER, *Washington, D.C.*
MARY REED, *Benton Harbor, Mich.*

Ex officio members:
EUGENE P. SMITH, *Detroit, Mich.*
MYRL H. AHRENDT, *Washington, D.C.*

Chairman:
MARY C. ROGERS, *Westfield, N.J.*

"School districts are increasingly offering summer schools, with emphasis on acceleration as well as remedial work."—*From "Schools in Our Democracy," Office of Education, U.S. Department of Health, Education, and Welfare.*

Analysis of membership growth, members and subscribers, May 1960 to May 1961

	May 1960	*May 1961*		*Goals and per cent of goals reached*	
		Individuals	*Totals*	*40,000 Goal*	*Per cent*
Alabama	316	246	373	451	83
Alaska	19	13	24	32	75
Arizona	339	367	462	160	289
Arkansas	227	206	283	456	62
California	2,455	1,801	2,972	2,344	123
Colorado	402	400	508	480	106
Connecticut	503	390	561	608	92
Delaware	111	92	119	187	64
District of Columbia	239	232	275	491	56
Florida	740	638	847	936	90
Georgia	527	296	638	496	129
Hawaii	137	121	171	121	141
Idaho	39	27	41	48	85
Illinois	2,157	2,478	3,178	3,168	100
Indiana	916	821	1,034	1,443	72
Iowa	522	440	643	787	82
Kansas	513	495	598	827	72
Kentucky	240	210	261	328	80
Louisiana	518	333	520	736	71
Maine	137	118	145	184	79
Maryland	529	483	690	760	91
Massachusetts	867	735	1,004	1,171	86
Michigan	1,852	1,255	2,067	1,600	129
Minnesota	889	711	938	1,068	88
Mississippi	209	198	240	328	73
Missouri	549	461	611	880	69
Montana	148	117	152	160	95
Nebraska	268	220	289	440	66
Nevada	44	44	57	35	163
New Hampshire	128	109	146	171	85
New Jersey	1,152	1,000	1,387	1,496	93
New Mexico	166	113	182	235	77
New York	2,893	2,325	3,221	3,147	102
North Carolina	459	282	566	680	83
North Dakota	127	126	159	115	138
Ohio	1,345	1,340	1,689	1,971	86
Oklahoma	472	418	614	632	97
Oregon	437	330	509	355	143
Pennsylvania	1,896	1,731	2,297	2,451	94
Rhode Island	144	125	169	187	90
South Carolina	207	172	256	363	71
South Dakota	90	60	95	96	99
Tennessee	339	277	372	656	57
Texas	1,425	1,093	1,543	1,792	86
Utah	144	121	174	128	136
Vermont	71	64	80	107	75
Virginia	519	469	687	952	72
Washington	511	320	534	608	88
West Virginia	183	149	180	419	43
Wisconsin	734	645	932	1,120	83
Wyoming	83	76	107	96	111
TOTALS	29,937	25,293	35,600	38,502	92
U.S. Possessions	36	46	61	64	95
Canada	670	554	906	536	169
Foreign	551	321	657	904	73
GRAND TOTALS	31,194	26,214	37,224	40,006	93

States relatively near the 40,000 goal

	Membership increase May 1960–May 1961	*Nearness to 40,000 goal*
Alabama	57	− 78
Alaska	5	− 8
Connecticut	58	− 47
Florida	107	− 89
Idaho	2	− 7
Iowa	121	−144
Maryland	161	− 70
Massachusetts	137	−167
Montana	4	− 8
New Hampshire	18	− 25
New Jersey	235	−109
North Carolina	107	−114
Ohio	344	−282
Oklahoma	142	− 18
Pennsylvania	401	−154
Rhode Island	25	− 18
South Dakota	5	− 1
Vermont	9	− 27
Wisconsin	198	−188
U.S. Possessions	25	− 3

States and territories having reached their goals or gone beyond

(Based on 40,000 goal)

Arizona	289%	North Dakota	138%	Wyoming	111%
Canada	169%	Utah	136%	Colorado	106%
Nevada	163%	Georgia	129%	New York	102%
Oregon	143%	Michigan	129%	Illinois	100%
Hawaii	141%	California	123%		

Membership achievement 90%-99%

(Based on 40,000 goals)

South Dakota	99%	Pennsylvania	94%	Maryland	91%
Oklahoma	97%	New Jersey	93%	Florida	90%
Montana	95%	Connecticut	92%	Rhode Island	90%
U.S. Possessions	95%				

Membership achievement 85%-89%

(Based on 40,000 goals)

Minnesota	88%	Ohio	86%	Idaho	85%
Washington	88%	Texas	86%	New Hampshire	85%
Massachusetts	86%				

Leaders in membership totals

(Including subscriptions)

New York	3,221	Ohio	1,689	Minnesota	938
Illinois	3,178	Texas	1,543	Wisconsin	932
California	2,972	New Jersey	1,387	Canada	906
Pennsylvania	2,297	Indiana	1,034	Florida	847
Michigan	2,067	Massachusetts	1,004	Maryland	690

Leaders in membership totals

(Not including subscriptions)

Illinois	2,478	Michigan	1,255	Minnesota	711
New York	2,325	Texas	1,093	Wisconsin	645
California	1,801	New Jersey	1,000	Florida	638
Pennsylvania	1,731	Indiana	821	Canada	554
Ohio	1,340	Massachusetts	735	Kansas	495

Leaders in membership growth

(Including subscriptions)

Illinois	1,021	Canada	236	Maryland	161
California	517	New Jersey	235	Oklahoma	142
Pennsylvania	401	Michigan	215	Massachusetts	137
Ohio	344	Wisconsin	198	Arizona	123
New York	328	Virginia	168	Iowa	121

Leaders in per cent of growth

(Compared with May, 1960, totals)

U.S. Possessions	Nevada	Colorado
Illinois	Oklahoma	Ohio
Arizona	Wyoming	Arkansas
Canada	Wisconsin	Hawaii
Virginia	Alaska	North Dakota
Maryland		

States and territories with continuous growth

(Since May, 1960)

Alabama	Kentucky	Oklahoma
Arizona	Massachusetts	U.S. Possessions
Delaware	Michigan	Vermont
Georgia	Ohio	

Membership analysis by territory

	May, 1960		*May 1, 1961*	
	Membership and per cent toward goal of 25,000		*Membership and per cent toward goal of 40,000*	
Pearl Bond Alabama, Arizona, Louisiana, Mississippi, Nevada, New Mexico, Texas	3,017	129%	3,377	90%
Janet Height New England States, Florida, Georgia	3,117	129%	3,590	93%
Lucille Houston Illinois, Iowa, Minnesota, Missouri, North Dakota, South Dakota, Wisconsin	5,068	112%	6,556	91%
Harold J. Hunt Alaska, California, Hawaii, Idaho, Montana, Oregon, Washington	3,746	163%	4,403	120%
Florence Ingham Arkansas, Colorado, Kansas, Nebraska, Oklahoma, Utah, Wyoming	2,109	110%	2,573	84%
Faith Novinger Delaware, District of Columbia, Maryland, New Jersey, New York, North Carolina, Pennsylvania, South Carolina	7,486	125%	8,811	92%
Mary Reed Indiana, Kentucky, Michigan, Ohio, Ontario, Tennessee, Virginia, West Virginia	6,064	123%	7,196	91%

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least two months prior to the month in which the issue appears, to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D.C.

NCTM convention dates

FORTIETH ANNUAL MEETING

April 16–18, 1962
Jack Tar Hotel, San Francisco, California
Kenneth C. Skeen, 3355 Cowell Road, Concord, California

JOINT MEETING WITH NEA

July 4, 1962
Denver, Colorado
M. H. Ahrendt, 1201 Sixteenth Street, N.W., Washington 6, D.C.

TWENTY-SECOND SUMMER MEETING

August 23–25, 1962
University of Wisconsin, Madison, Wisconsin
H. Van Engen, School of Education, University of Wisconsin, Madison 6, Wisconsin

Other professional dates

Arkansas Council of Teachers of Mathematics

October 6–7, 1961
Arkansas State Teachers College, Conway, Arkansas
Mrs. R. H. Jones, 1229 Dyer, Malvern, Arkansas

Women's Mathematics Club of Chicago and Vicinity

October 7, 1961
Wedgwood Room, Marshall Field & Company, Chicago, Illinois
Dr. Ruth Ballard, University of Illinois, Navy Pier, Chicago 11, Illinois

Illinois Council of Teachers of Mathematics

October 13–14, 1961
University of Illinois, Champaign-Urbana, Illinois
T. E. Rine, Illinois State Normal University, Normal, Illinois

Maryland State Teachers' Association, Mathematics Section

October 20, 1961
Edmondson High School, Edmondson Village, Maryland
W. Edwin Freeny, Woodlawn High School, Baltimore 7, Maryland

Washington State Mathematics Council

October 27–28, 1961
University of Puget Sound
Professor Ed Goman, 1500 North Warner, Tacoma, Washington

Women's Mathematics Club of Chicago and Vicinity

November 4, 1961
English Room, Marshall Field & Company, Chicago, Illinois
Dr. Ruth Ballard, University of Illinois, Navy Pier, Chicago 11, Illinois

Georgia Mathematics Council

November 10–11, 1961
Rock Eagle State Park, Eatonton, Georgia
Martha Rogers, 2908 Macon Road, Columbus, Georgia

Women's Mathematics Club of Chicago and Vicinity

December 2, 1961
Henrici's Restaurant, 71 W. Randolph, Chicago, Illinois
Dr. Ruth Ballard, University of Illinois, Navy Pier, Chicago 11, Illinois

"Hundreds of schools are revising their courses of study in mathematics, science, foreign languages, English, and social studies.—*From "Schools in Our Democracy," Office of Education, U.S. Department of Health, Education, and Welfare.*

Candidates for offices in The National Council of Teachers of Mathematics

The following slate of officers was approved by the Board of Directors at the Business Meeting held in Toronto August, 1961.

President

Frank B. Allen, Illinois
Donovan Johnson, Minnesota

Vice-President, Secondary Level

Eugene P. Smith, Michigan
Florence Elder, New York

Vice-President, Elementary Level

Edwina Deans, Washington, D.C.
Marguerite Brydegaard, California

Directors

William W. Matson, Oregon
Edward Begle, California
Burton W. Jones, Colorado
Isabelle Rucker, Virginia
Clarence H. Heinke, Ohio
Emil J. Berger, Minnesota

NCTM receives $18,000 IBM grant to enrich high school math program

International Business Machines Corporation (IBM) of New York City has made a grant of $18,000 to the National Council of Teachers of Mathematics (NCTM) for the preparation and distribution of a series of publications on digital computers to enrich the high school mathematics program.

This project is not an attempt to make computer operators of high school students. The emphasis will be on the fundamental mathematics involved in computing, and the purpose is to motivate the learning of mathematics. The writing of programs, selected to give students insight into mathematical concepts, will be stressed.

The first series of teaching units, usable either as text or enrichment material, will consist of separate publications for teachers and students. They will emphasize logical diagrams applied to problem solving, which will be illustrated and discussed at a wide range of levels. Typical topics are various types of equations, matrices, interpolation, numerical integration, and the like; computation with approximate data; explanation of the ways in which the machine handles data with discussions of input, output, memory, computer parts, and machine language; computations involving repeated operations; and applications to various fields and problems.

A second type of publication will cover careers in data processing. The role of mathematics throughout the entire field of computer usage will be emphasized, and will deal with the uses of mathematics not only in programming, but also in sales, design, maintenance, customer applications, and others.

The third publication will be designed as a guide for both teachers and students. It will include suggestions for the use of the material developed in the other booklets, comments from teachers experienced in the use of computer mathematics, bibliographies, and reference material.

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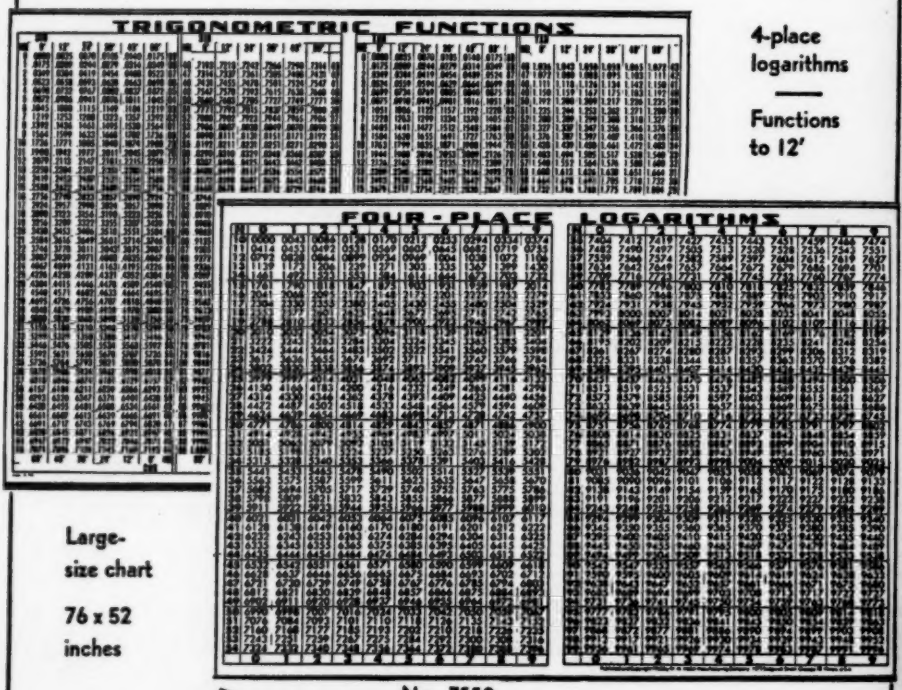

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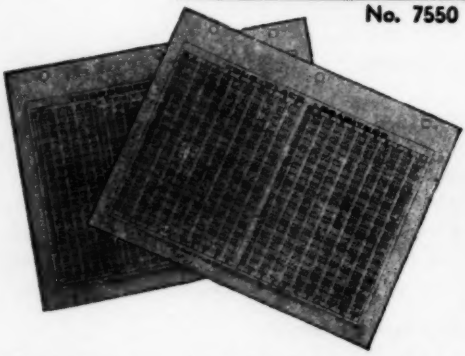

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